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STABILITY OF RECTANGULAR PLATES WITH LONGITUDINAL OR

TRANSVERSE STIFFENERS UNDER UNIFORM COMPRESSION

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I. INTRODUCTION

The proper application of stiffeners, i.e., stiffen-
ing ribs fixed to a plate, leads to an increase of the
ιg strength of rectangular plates. In calculating
.ed plates, we have to distinguish between:

Plates with large spacing of the stiffeners in
he bending stiffnesses of the plate and ribs appear
ely in the calculation, and

Plates with small stiffener spacings for which
ding stiffness of plate and stiffeners in the direc-
the stiffeners can be combined to a new bending
ss, provided the stiffeners all have the same cross
. In general, we are allowed to treat such plates
as orthotropic plates.

The first investigation on the stability of plates,
corresponding to 1) above, was made by Timoshenko (refer-
ence 2), who calculated the buckling stress of plates with
one to three longitudinal or transverse stiffeners with
equal spacings. He considered hinged plate edges and two
loadings, uniform compression and pure shear. It is well
to note here that in the following discussion those stif-
feners in the direction of the normal loading are called
longitudinal, and those perpendicular to the direction of
loading are called transverse.

Timoshenko uses the energy method for solution, a
method which was applied by Bryan (reference 3) in his
classical work on the buckling of a rectangular plate, and
which was later on - taking into consideration the mass
forces - exactly proven by Reissner (reference 4). Recent-
ly the stiffened plate with one longitudinal stiffener in

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*"Stabilität gleichmässig gedrückter Rechteckplatten mit
    Längs- oder Quersteifen." Ingenieur-Archiv, vol. 8,
    no. 2, 1937, pp. 117-150.

the middle of the compression field, i.e., the middle of
the plate, stressed by pure bending, compression, or shear,
was considered by Chwalla (references 5 and 6), and the
problem was solved also with the aid of the energy method.

In the application of the energy method, the wave form
is assumed to be represented by a series which satisfies
the boundary conditions, and the coefficients of which are
determined by minimum energy considerations. The more ex-
actly the assumed shape of the wave pattern agrees with
the actual pattern, the less terms in the summation are
necessary for a sufficiently accurate calculation of the
buckling stresses. Exact buckling conditions in a finished
form cannot be developed with this method. However, for
some cases, concerning the loading and the reactions of the
plate, we have complete solutions of the differential equa-
tion and in these cases the buckling conditions can be rep-
resented exactly. As the stiffened plate consists of a num-
ber of nonstiffened strips which are connected with each
other along the stiffeners, the solutions of the differen-
tial equation for the nonstiffened plate can be accordingly
applied to the stiffened plate.

For the unstiffened plate Timoshenko (reference 7),
Reissner (reference 8), and in a more complete manner,
Chwalla (reference 9) have set up the buckling conditions
for uniform compression in one direction, the loaded edges
being hinged with optional support of the longitudinal
edges; with the aid of the complete solution of the differ-
ential equation. The solutions for supported transverse
edges and hinged longitudinal edges originate from
Schleicher (reference 10).

With shear stresses, complete solutions are known only
for the infinitely long strip. The fundamental investiga-
tion for this is the work of Southwell and Skan (reference
11), in which pure shear stress with hinged and fixed lon-
gitudinal edges is investigated. Schmieden (reference 12)
develops solutions for combined shear and compression of
the infinitely long strip.

In the present paper, the complete buckling conditions
of stiffened plates are being developed for uniform compres-
sion. We shall treat plates with one or two longitudinal
or transverse stiffeners at any point, discuss the buckling
conditions, and evaluate them for different cases.

For the special case with any number of longitudinal stiffeners with equal dimensions and with equal spacings, Lokshin (reference 13) has calculated the buckling conditions. However, as we shall prove in the following, these buckling conditions are not complete.

II. PLATE WITH LONGITUDINAL STIFFENERS

1. General principles.- The rectangular plate with the length-width ratio $\alpha = a/b$ is equipped with longitudinal stiffeners at the points $y = b_1$, $y = b_1 + b_2$, etc., by which it is divided into several nonstiffened areas with the ratios $\alpha_1 = a/b_1$, $\alpha_2 = a/b_2 \ldots \alpha_r = a/b_r$ (fig. 1). If the plate is loaded with uniform normal stresses σ_x at the edges $x = 0$ and $x = a$, the buckling deformation $w = w(x,y)$ of the plate middle area inside of any field i satisfies the differential equation:

$$\frac{\partial^4 w_i}{\partial x^4} + 2 \frac{\partial^4 w_i}{\partial x^2 \partial y^2} + \frac{\partial^4 w_i}{\partial y^4} - \frac{t\sigma_x}{D} \frac{\partial^2 w_i}{\partial x^2} \equiv \Delta\Delta w_i - \frac{t\sigma_x}{D} \frac{\partial^2 w_i}{\partial x^2} = 0 \qquad (1)$$

in which
$$D = \frac{E\,t^3}{12(1 - \mu^2)}$$

is the stiffness and t is the thickness of the plate. Considering the ratios

$$\xi = \frac{x}{b_1}, \qquad \eta = \frac{y}{b_1}$$

$$\varphi_1 = - \frac{\sigma_x}{\sigma_{1e}} \left(\sigma_{1c} = \frac{\pi^2 D}{b_1^2 t}\right. = \text{(Euler buckling stress) (ref-}$$

erence 14). In reference to the width b_1 of the first plate field, the differential equation (1) changes into

$$\Delta \Delta w_i (\xi,\eta) + \pi^2 \varphi_1 \frac{\partial^2 w_i}{\partial \xi^2} = 0 \qquad (2)$$

Under the assumption of hinged mounting, $(w = \Delta w = 0)$ at the borders $\xi = 0$ and $\xi = \alpha_1$, which is valid for the following considerations, the differential equation (2) is

satisfied by the equation*

$$w_i = \sum_1^\infty m_i \, Y_{m_i}(\eta) \sin v_{m_i} \, \xi \quad \text{with} \quad v_{m_i} = \frac{m_i \pi}{\alpha_1} \quad (m_i = 1, 2, 3 \ldots)$$

(3)

Substituting this solution in equation (2), we obtain for the function $Y_{m_i}(\eta)$, which only depends upon η, for every value m_i, the ordinary differential equation

$$Y_{m_i}^{IV} - 2 v_{m_i}^2 \, Y_{m_i}^{II} + v_{m_i}^2 \, (v_{m_i}^2 - \pi^2 \, \varphi_1) \, Y_{m_i} = 0 \quad .$$

(4)

the solution of which is

$$Y_{m_i} = A_i \sinh \kappa_{1i} \, \eta + B_i \cosh \kappa_{1i} \, \eta +$$
$$+ \, C_i \sin \kappa_{2i} \, \eta + D_i \cos \kappa_{2i} \, \eta$$

(5)

in which

$$\kappa_{\genfrac{}{}{0pt}{}{1i}{2i}} = \sqrt{v_{m_i} \, (\pi \sqrt{\varphi_1} \pm v_{m_i})}$$

For determining the constants A_i to D_i,** we have homogeneous equations at our disposal; the solutions $w \neq 0$ are only for special values φ_1, the so-called buckling values k_1, with the critical buckling stresses $\sigma_k = k_1 \, \sigma_{1e}$. Every term of the solution (3) satisfies the boundary conditions at the borders $\xi = 0$ and $\xi = \alpha_1$; therefore for each value m_i buckling values k_1 can be calculated, the buckling areas of which run in a sine curve in the ξ-direction with m_i half-waves. As every two adjoining fields are continuously connected along the longitudinal stiffeners, it is necessary for obtaining the buckling, sine-shaped in the ξ-direction, that the number of half-waves in both fields – and therefore in all fields – be the same. For this reason, we can put

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*This expression, by which the buckle in the ξ-direction is assumed to be a sine-curve, was used by Timoshenko, Reissner, and others for problems of stability; actually, it is even older and was formerly used in probelms for bending of rectangular plates.

**The constants D_i with the subscript i must not be confused with the stiffness of the plate D.

$$m_i = m, \qquad \nu_{m_i} = \nu_m, \qquad \kappa_{1_i \atop 2_i} = \kappa_{1 \atop 2}$$

The values κ_1 and κ_2 are real for the actually occurring cases, since the buckling stresses of the stiffened plate, whose borders $\eta = 0$ and $\eta = b/b_1 = \beta$ are generally strengthened by some form of support, are greater than the minimum values of the real solutions $(\pi\sqrt{k_1} = \nu_m)$ which give the Euler critical stress

$$\sigma_k = \frac{m^2 \pi^2 D}{a^2 t}$$

of the strip with the length a, whose longitudinal edges are under uniform compression.

As will be shown in the next section, we have, for determining the $4r$ constants A_i, B_i, C_i, D_i ($i = 1, 2, \dots r$) four boundary conditions at the borders $\eta = 0$ and and $\eta = b/b_1 = \beta$ and, furthermore, $4(r - 1)$ transitional conditions at the stiffeners which form a system of $4r$ homogeneous equations. We obtain the buckling condition with the aid of these equations by putting the determinant of the denominator equal to zero. Only the minimum values of the roots of the buckling equations are of interest, the other roots representing higher buckling values.

The buckling values k with respect to the total width b of the plate, i.e., (reference 14) $k = \sigma_x/\sigma_e$, are calculated from k_1:

$$k = k_1 \frac{\sigma_{1e}}{\sigma_e} = k_1 \left(\frac{b}{b_1}\right)^2 = k_1 \beta^2$$

2. <u>Boundary and transitional conditions</u>.- At the borders $\eta = 0$ and $\eta = \beta$, the plate is generally connected to edge supports. In many cases these supports have relatively large and compact cross sections, so that a certain elastic mounting of the plate with them is given. This mounting should be taken into account in the general boundary conditions. (See reference 9.)

For the stiffeners and supports 0 to r, respectively, we introduce the following notations:

F_0 to F_r,   cross sections

$E\ J_{\eta o}$ to $E\ J_{\eta r}$, bending stiffness about η axis

$G\ T_0$ to $G\ T_r$, torsional stiffness*

Along the border line $\eta = 0$ between the field 1
and the support 0, there acts as an internal force the
bending moment

$$m_\eta = -\frac{D}{b_1{}^2}\left(\frac{\partial^2 w_1}{\partial \eta^2} + \mu\,\frac{\partial^2 w_1}{\partial \xi^2}\right) \tag{6}$$

and the reaction forces

$$q_\eta = -\frac{D}{b_1{}^3}\left(\frac{\partial^3 w_1}{\partial \eta^3} + (2-\mu)\,\frac{\partial^3 w_1}{\partial \xi^2 \partial \eta}\right) \tag{7}$$

of the plate field 1, which, if we consider the plate cut
off along the support, are to be applied as external loads
to the support (fig. 2).

The bending moments m_η create a torsional stress in
the support. The change of the total torsional moment M_ξ
acting at the point ξ is therefore

$$\frac{d\,M_\xi}{b_1 d\xi} = m_\eta$$

Under the assumption that the change of slope of the
cross sections can be neglected in the torsion considera-
tion, which applies exactly only for circular and ring-
shaped plates, but can be approximately assumed for other
shapes, the mutual twisting $d\vartheta$ of the stiffening element
$b_1\,d\xi$ is:

$$d\vartheta = \frac{M_\xi}{G\ T_0}\ b_1\ d\xi$$

and from this follows:

$$M_\xi = G\ T_0\ \frac{d\vartheta}{b_1 d\xi}$$

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*The values T_0 to T_r may be taken from the results of
Weber's work (reference 15) or from that of Förster (ref-
erence 16).

On account of the continuous connection of stiffener and plate, the twisting of the stiffener is equal to the slope of the plate, i.e.,

$$\vartheta \approx \tan \vartheta = \frac{\partial w_1}{b_1 \partial \eta}$$

therefore

$$m\eta = G \, T_0 \, \frac{d^2 \vartheta}{b_1^2 \, d\xi^2} = G \, T_0 \, \frac{\partial^3 w_1}{b_1^3 \, \partial \xi^2 \, \partial \eta}$$

so that the first boundary condition at the point $\eta = 0$ is

$$- \frac{D}{b_1^2} \left(\frac{\partial^2 w_1}{\partial \eta^2} + \mu \, \frac{\partial^2 w_1}{\partial \xi^2} \right) = G \, T_0 \, \frac{\partial^3 w_1}{b_1^3 \, \partial \xi^2 \partial \eta} \tag{8}$$

The reactions $q\eta$ of the plate field 1 create in the support a bending moment $M\eta$ about the η-axis, assuming the border of the plate to be at the shear center of the support. If, at the same time, the support is acted on by the compression stresses σ_{xk}, then, at its ends, we have the compression forces

$$P_0 = \sigma_{xk} \, F_0 = k_1 \, \sigma_{1e} \, F_0$$

which at the point ξ induce the bending moment $P_0 \, w_1$. For the bending of the stiffener which, on account of the continuous connection between it and the sheet, agrees with the deflections w_1 for $\eta = 0$, we have therefore the differential equation

$$E \, J\eta_0 \, \frac{\partial^2 w_1}{b_1^2 \, \partial \xi^2} = - M = - M\eta - P_0 \, w_1$$

From this, by differentiating twice with respect to ξ a second boundary condition for $\eta = 0$ is obtained:

$$EJ\eta_0 \, \frac{\partial^4 w_1}{b_1^2 \, \partial \xi^4} = - \frac{D}{b_1} \left(\frac{\partial^3 w_1}{\partial \eta^3} + (2-\mu) \, \frac{\partial^3 w_1}{\partial \xi^2 \partial \eta} \right) - k_1 \, \sigma_{1e} \, F_0 \, \frac{\partial^2 w_1}{\partial \xi^2} \tag{9}$$

For the support r, i.e., for $\eta = \beta$, the corresponding

boundary conditions are

$$\frac{D}{b_1^2}\left(\frac{\partial^2 w_r}{\partial \eta^2} + \mu \frac{\partial^2 w_r}{\partial \xi^2}\right) = G\ T_r\ \frac{\partial^3 w_r}{b_1^3\ \partial \xi^2 \partial \eta} \qquad (10)$$

and

$$EJ_{\eta r}\ \frac{\partial^4 w_r}{b_1^2\ \partial \xi^4} = \frac{D}{b_1}\left(\frac{\partial^3 w_r}{\partial \eta^3} + (2-\mu)\frac{\partial^3 w_r}{\partial \xi^2 \partial \eta}\right) - k_1\ \sigma_{1e}\ F_0\ \frac{\partial^2 w_r}{\partial \xi^2} \qquad (11)$$

Between the adjoining fields i and i + 1, there is the stiffener i with the cross section F_i and the moment of inertia $J_{\eta i}$ about the η-axis. At the stiffener, the conditions of continuous connection to the sheet have to be first satisfied. Therefore, for

$$\eta = \frac{b_1 + b_2 + \ldots + b_i}{b_1}$$

we obtain

$$w_i = w_{i+1} \qquad (12)$$

and

$$\frac{\partial w_i}{\partial \eta} = \frac{\partial w_{i+1}}{\partial \eta} \qquad (13)$$

Since the cross sections of the stiffeners generally have I, L, or Z forms, indicating no great torsional stiffness, we may neglect the torsional stress which occurs in the stiffeners due to the bending of the plate. The experiments of Erlemann (reference 17) justify this assumption. The moments m_η of the fields i and i+1 at the stiffener are therefore equal, i.e.,

$$- \frac{D}{b_1^2}\left(\frac{\partial^2 w_i}{\partial \eta^2} + \mu \frac{\partial^2 w_i}{\partial \xi^2}\right) = - \frac{D}{b_1^2}\left(\frac{\partial^2 w_{i+1}}{\partial \eta^2} + \mu \frac{\partial^2 w_{i+1}}{\partial \xi^2}\right) \quad (14a)$$

Along the stiffener, we have, as a further condition of the continuity

$$\frac{\partial^2 w_i}{\partial \xi^2} = \frac{\partial^2 w_{i+1}}{\partial \xi^2} \qquad (14b)$$

so that the transitional condition (14a) becomes

$$\frac{\partial^2 w_i}{\partial \eta^2} = \frac{\partial^2 w_{i+1}}{\partial \eta^2} \tag{14}$$

Considering the fields i and i+1 as cut off at the stiffener, a transitional condition corresponding to boundary condition (9) applies, but instead of the reaction force of the border field, the difference between the reactions of the fields i and i+1, adjoining at the stiffener, have to be introduced. Therefore,

$$EJ_{\eta i} \frac{\partial^4 w_i}{b_1^2 \partial \xi^4} = -\frac{D}{b_1} \left(\frac{\partial^3 w_{i+1}}{\partial \eta^3} + (2-\mu) \frac{\partial^3 w_{i+1}}{\partial \xi^2 \partial \eta} - \frac{\partial^3 w_i}{\partial \eta^3} - \right.$$

$$\left. - (2-\mu) \frac{\partial^3 w_i}{\partial \xi^2 \partial \eta} \right) - k_1 \sigma_{1e} F_i \frac{\partial^2 w_i}{\partial \xi^2} \tag{15a}$$

Considering equations (13) and (14b), we have at the stiffener,

$$\frac{\partial^3 w_i}{\partial \xi^2 \partial \eta} = \frac{\partial^3 w_{i+1}}{\partial \xi^2 \partial \eta} \tag{15b}$$

so that the transitional condition (15a) is simplified to

$$EJ_{\eta i} \frac{\partial^4 w_i}{b_1^2 \partial \xi^4} = -\frac{D}{b_1} \left(\frac{\partial^3 w_{i+1}}{\partial \eta^3} - \frac{\partial^3 w_i}{\partial \eta^3} \right) - k_1 \sigma_{1e} F_i \frac{\partial^2 w_i}{\partial \xi^2} \tag{15}$$

In the term on the left side and the last term of the right side, the index i of w may be replaced by i + 1, as it makes no difference to which of the adjoining fields the deflections of the stiffener are referenced. Equation (15) applies exactly only to the symmetrically connected stiffener, the neutral axis of which coincides with that of the plate. In many cases, the stiffener will be fastened only on one side of the plate. In this case, according to the proposition of Timoshenko (reference 2), the moment of inertia $J_{\eta i}$ must be referred to the axis which lies in the connecting surface between the stiffener and plate. The additional stresses in the plate which arise from this condition are not taken into consideration. They fade very fast along the effective width in the η-direction, according to statements by Chwalla (reference 18).

Table 1

	Gl.	a A_1	b B_1	c C_1	d D_1	e A_2	f B_2	g C_2	h D_2
1	(8')	$-\Psi_0\varkappa_1$	$+e_1$	$-\Psi_0\varkappa_2$	$-e_2$				
2	(9')	$-\varkappa_1 c_1$	$-\Phi_0$	$+\varkappa_2 c_2$	$-\Phi_0$				
3	(12')	$\otimes\ \mathrm{Sin}\,\varkappa_1$	$\oplus\ \mathrm{Cof}\,\varkappa_1$	$\sin\varkappa_2$	$\cos\varkappa_2$	$-\mathrm{Sin}\,\varkappa_1$	$-\mathrm{Cof}\,\varkappa_1$	$-\sin\varkappa_2$	$-\cos\varkappa_2$
4	(14')	$\varkappa_1^2\,\mathrm{Sin}\,\varkappa_1$	$\varkappa_1^2\,\mathrm{Cof}\,\varkappa_1$	$-\varkappa_2^2\sin\varkappa_2$	$-\varkappa_2^2\cos\varkappa_2$	$-\varkappa_1^2\,\mathrm{Sin}\,\varkappa_1$	$-\varkappa_1^2\,\mathrm{Cof}\,\varkappa_1$	$+\varkappa_2^2\sin\varkappa_2$	$+\varkappa_2^2\cos\varkappa_2$
5	(13')	$\varkappa_1\,\mathrm{Cof}\,\varkappa_1$	$\varkappa_1\,\mathrm{Sin}\,\varkappa_1$	$\varkappa_2\cos\varkappa_2$	$-\varkappa_2\sin\varkappa_2$	$-\varkappa_1\,\mathrm{Cof}\,\varkappa_1$	$-\varkappa_1\,\mathrm{Sin}\,\varkappa_1$	$-\varkappa_2\cos\varkappa_2$	$+\varkappa_2\sin\varkappa_2$
6	(15')	$-\varkappa_1^3\,\mathrm{Cof}\,\varkappa_1$ $+\Phi_1\,\mathrm{Sin}\,\varkappa_1$	$-\varkappa_1^3\,\mathrm{Sin}\,\varkappa_1$ $+\Phi_1\,\mathrm{Cof}\,\varkappa_1$	$+\varkappa_2^3\cos\varkappa_2$ $+\Phi_1\sin\varkappa_2$	$-\varkappa_2^3\sin\varkappa_2$ $+\Phi_1\cos\varkappa_2$	$+\varkappa_1^3\,\mathrm{Cof}\,\varkappa_1$	$+\varkappa_1^3\,\mathrm{Sin}\,\varkappa_1$	$-\varkappa_2^3\cos\varkappa_2$	$+\varkappa_2^3\sin\varkappa_2$
7	(10')					$e_1\,\mathrm{Sin}\,\beta\varkappa_1$ $+\Psi_2\varkappa_1\,\mathrm{Cof}\,\beta\varkappa_1$	$e_1\,\mathrm{Cof}\,\beta\varkappa_1$ $+\Psi_2\varkappa_1\,\mathrm{Sin}\,\beta\varkappa_1$	$-e_2\sin\beta\varkappa_2$ $+\Psi_2\varkappa_2\cos\beta\varkappa_2$	$-e_2\cos\beta\varkappa_2$ $-\Psi_2\varkappa_2\sin\beta\varkappa_2$
8	(11')					$\varkappa_1 c_1\,\mathrm{Cof}\,\beta\varkappa_1$ $-\Phi_2\,\mathrm{Sin}\,\beta\varkappa_1$	$\varkappa_1 c_1\,\mathrm{Sin}\,\beta\varkappa_1$ $-\Phi_2\,\mathrm{Cof}\,\beta\varkappa_1$	$-\varkappa_2 c_2\cos\beta\varkappa_2$ $-\Phi_2\sin\beta\varkappa_2$	$\varkappa_2 c_2\sin\beta\varkappa_2$ $-\Phi_2\cos\beta\varkappa_2$

$\otimes$ = sinh

$\oplus$ = cosh

In the following, we take the ratios between bending stiffnesses, dependent upon the cross sections of the stiffeners, and those of the plate from Timoshenko (reference 2):

$$\gamma_{1i} = \frac{EJ_{\eta i}}{b_1 D}, \qquad\qquad \gamma_i = \frac{EJ_{\eta i}}{bD} = \gamma_{1i}\frac{b_1}{b},$$

$$\delta_{1i} = \frac{F_i}{b_1 t}, \qquad\qquad \delta_i = \frac{F_i}{bt} = \delta_{1i}\frac{b_1}{b}$$

and, to abbreviate, we place

$$\Phi_i = \frac{EJ_{\eta i}}{b_1 D} \nu^4 - k_1 \sigma_{1e} \frac{F_i b_1}{D} \nu^2 = \gamma_{1i}\nu^4 - \pi^2 k_1 \delta_{1i}\nu^2$$

and

$$\psi_i = \frac{G\, T_i}{b_1 D}\nu^2$$

After introducing the solution (3) in the boundary and transitional conditions (8) to (15), the following equations, independent of ξ are obtained, which serve to determine the constants A to D in the functions Y. In these equations the index m (m = number of half-waves in the ξ-direction) is omitted for reasons of simplification.

$$\left.\begin{array}{l} Y_1'' - \psi_0\, Y_1' - \mu\,\nu^2\, Y_1 = 0 \\[2mm] -\, Y_1''' + \nu^2\,(2-\mu)\, Y_1' - \Phi_0\, Y_1 = 0 \end{array}\right\} \quad \text{for}\quad \eta = 0 \qquad\qquad \begin{array}{l}(8\,') \\[2mm] (9\,')\end{array}$$

$$\left.\begin{array}{l} Y_r'' + \psi_r\, Y_r' - \mu\,\nu^2\, Y_r = 0 \\[2mm] +\, Y_r''' - \nu^2\,(2-\mu)\, Y_r' - \Phi_r\, Y_r = 0 \end{array}\right\} \quad \text{for}\quad \eta = \beta \qquad\qquad \begin{array}{l}(10\,') \\[2mm] (11\,')\end{array}$$

$$\left.\begin{array}{l} Y_i - Y_{i+1} = 0 \\[2mm] Y_i' - Y_{i+1}' = 0 \\[2mm] Y_i'' - Y_{i+1}'' = 0 \\[2mm] -Y_i''' + Y_{i+1}''' + \Phi_i Y_i = 0 \end{array}\right\} \quad \text{for}\quad \eta = \frac{b_1 + b_2 + \ldots + b_i}{b_1} \qquad\qquad \begin{array}{l}(12\,') \\[2mm] (13\,') \\[2mm] (14\,') \\[2mm] (15\,')\end{array}$$

3. *Plate with one longitudinal stiffener.*— In the case of one longitudinal stiffener, $r = 2$. With the solutions,

$$w_1 = Y_1 \sin \nu \xi, \quad Y_1 = A_1 \sinh \kappa_1 \eta + B_1 \cosh \kappa_1 \eta +$$

$$+ C_1 \sin \kappa_2 \eta + D_1 \cos \kappa_2 \eta,$$

$$w_2 = Y_2 \sin \nu \xi, \quad Y_2 = A_2 \sinh \kappa_1 \eta + B_2 \cosh \kappa_1 \eta +$$

$$+ C_2 \sin \kappa_2 \eta + D_2 \cos \kappa_2 \eta$$

we obtain from (8') to (15') a system of homogeneous equations, the coefficients of which, with the abbreviations

$$e_1 = \kappa_1^2 - \mu \nu^2, \qquad e_2 = \kappa_2^2 + \mu \nu^2$$

$$c_1 = \kappa_1^2 - \nu^2(2 - \mu), \qquad c_2 = \kappa_2^2 + \nu^2(2 - \mu)$$

$$(e_1 + e_2 = c_1 + c_2 = \kappa_1^2 + \kappa_2^2)$$

are shown in table 1. This table is, at the same time, the denominator determinant to be solved, the lines of which are denoted by the numbers 1 to 8, and the columns by the letters a to h.

a) *Solution of the determinant and the general buckling conditions.*— Solving this determinant, we denote the subdeterminants as follows:

$$\begin{vmatrix} 1, & 2 \\ a, & b \end{vmatrix}$$

is the subdeterminant which is obtained from the complete determinant by canceling the lines 1 and 2 and the columns a and b. The determinants marked with overlining are to be formed from the subdeterminants on the left side of the equation.

After eliminating the lines 1 and 2, the determinant becomes:

$$\left.\begin{array}{l} \kappa_1 (\psi_0 \Phi_0 + c_1 e_1) \begin{vmatrix} 1,2 \\ a,b \end{vmatrix} + \kappa_1 \kappa_2 \psi_0 (\kappa_1{}^2 + \kappa_2{}^2) \begin{vmatrix} 1,2 \\ a,c \end{vmatrix} + \\[2ex] + \kappa_1 (\psi_0 \Phi_0 - c_1 e_2) \begin{vmatrix} 1,2 \\ a,d \end{vmatrix} \\[2ex] + \kappa_2 (-\psi_0 \Phi_0 + c_2 e_1) \begin{vmatrix} 1,2 \\ b,c \end{vmatrix} + \Phi_0 (\kappa_1{}^2 + \kappa_2{}^2) \begin{vmatrix} 1,2 \\ b,d \end{vmatrix} + \\[2ex] + \kappa_2 (\psi_0 \Phi_0 + c_2 e_2) \begin{vmatrix} 1,2 \\ c,d \end{vmatrix} \end{array}\right\} \quad (16)$$

The lines 1 and 2 contain only the unknowns A_1 to D_1, the lines 7 and 8, only the unknowns A_2 to D_2. According to this, the lines 1 and 2 are independent of the lines 7 and 8, so that in the following elimination of the lines 7 and 8, the subdeterminants

$$\begin{vmatrix} 1, & 2 \\ a, & b \end{vmatrix}, \quad \cdots \quad \begin{vmatrix} 1, & 2 \\ c, & d \end{vmatrix}$$

with $p = a, b, c,$ and $q = b, c, d$ $(p \neq q)$ occurring in (16), may be generally denoted by

$$\begin{vmatrix} 1, & 2 \\ p, & q \end{vmatrix} = (- c_1 e_1 - \psi_2 \Phi_2) \kappa_1 \begin{vmatrix} 7, & 8 \\ e, & f \end{vmatrix}$$

$$+ [(\psi_2 \Phi_2 - c_1 e_2) \kappa_1 \cosh \beta \kappa_1 \sin \beta \kappa_2 +$$

$$+ (-\psi_2 \Phi_2 + c_2 e_1) \kappa_2 \sinh \beta \kappa_1 \cos \beta \kappa_2 +$$

$$+ \Phi_2 (\kappa_1{}^2 + \kappa_2{}^2) \sinh \beta \kappa_1 \sin \beta \kappa_2 +$$

$$+ \psi_2 \kappa_1 \kappa_2 (\kappa_1{}^2 + \kappa_2{}^2) \cosh \beta \kappa_1 \cos \beta \kappa_2] \begin{vmatrix} 7, & 8 \\ e, & g \end{vmatrix}$$

$$+ [(-\Psi_2 \Phi_2 + c_2 e_1) \varkappa_2 \mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 - \Phi_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 +$$

$$+ \Psi_2 \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 + (-\Psi_2 \Phi_2 + c_1 e_2) \varkappa_1 \mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2]\, \overline{\begin{vmatrix} 7,\ 8 \\ e,\ h \end{vmatrix}}$$

$$+ [(-\Psi_2 \Phi_2 + c_1 e_2) \varkappa_1 \mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 - \Psi_2 \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 -$$

$$- \Phi_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 + (\Psi_2 \Phi_2 - c_2 e_1) \varkappa_2 \mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2]\, \overline{\begin{vmatrix} 7,\ 8 \\ f,\ g \end{vmatrix}}$$

$$+ [-\Psi_2 \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 + (\Psi_2 \Phi_2 - c_1 e_2) \varkappa_1 \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 +$$

$$+ (\Psi_2 \cdot \Phi_2 - c_2 e_1) \varkappa_2 \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 + \Phi_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2]\, \overline{\begin{vmatrix} 7,\ 8 \\ f,\ h \end{vmatrix}}$$

$$+ (-c_2 e_2 - \Psi_2 \Phi_2) \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ g,\ h \end{vmatrix}}.$$

Separating the factors of Φ_2, ψ_2 and $\Phi_2 \cdot \psi_2$, we get

$$\begin{vmatrix} 1,\ 2 \\ p,\ q \end{vmatrix} = - \varkappa_1 c_1 e_1 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ f \end{vmatrix}} - \varkappa_2 c_2 e_2 \overline{\begin{vmatrix} 7,\ 8 \\ g,\ h \end{vmatrix}}$$

$$+ \varkappa_1 c_1 e_2 \left(- \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ g \end{vmatrix}} + \mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ h \end{vmatrix}} + \right.$$

$$\left. + \mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ g \end{vmatrix}} - \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ h \end{vmatrix}} \right)$$

$$+ \varkappa_2 c_2 e_1 \left(+ \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ g \end{vmatrix}} + \mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ h \end{vmatrix}} - \right.$$

$$\left. - \mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ g \end{vmatrix}} - \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ h \end{vmatrix}} \right)$$

$$+ \Phi_2 (\varkappa_1^2 + \varkappa_2^2) \left(\mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ g \end{vmatrix}} - \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ h \end{vmatrix}} - \right.$$

$$\left. - \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ g \end{vmatrix}} + \mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ h \end{vmatrix}} \right) \qquad (17)$$

$$+ \Psi_2 \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2) \left(\mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ g \end{vmatrix}} + \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ h \end{vmatrix}} - \right.$$

$$\left. - \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ g \end{vmatrix}} - \mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ h \end{vmatrix}} \right)$$

$$+ \Phi_2 \cdot \Psi_2 \left[\varkappa_1 \left(- \overline{\begin{vmatrix} 7,\ 8 \\ e,\ f \end{vmatrix}} + \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ g \end{vmatrix}} - \mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ h \end{vmatrix}} - \right. \right.$$

$$\left. - \mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ g \end{vmatrix}} + \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ h \end{vmatrix}} \right)$$

$$+ \varkappa_2 \left(- \overline{\begin{vmatrix} 7,\ 8 \\ g,\ h \end{vmatrix}} - \mathfrak{Sin}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ g \end{vmatrix}} - \mathfrak{Sin}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ e,\ h \end{vmatrix}} + \right.$$

$$\left. \left. + \mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ g \end{vmatrix}} + \mathfrak{Cof}\, \beta \varkappa_1 \sin \beta \varkappa_2 \overline{\begin{vmatrix} 7,\ 8 \\ f,\ h \end{vmatrix}} \right) \right].$$

The solution of the remaining subdeterminants with four columns gives the following values:

$$\begin{vmatrix} 1,\ 2,\ 7,\ 8 \\ a,\ b,\ e,\ f \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1,\ 2,\ 7,\ 8 \\ a,\ b,\ e,\ g \end{vmatrix} = - \Phi_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \varkappa_1 \cos \varkappa_2,$$

$$\begin{vmatrix} 1,\ 2,\ 7,\ 8 \\ a,\ b,\ e,\ h \end{vmatrix} = - \Phi_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \varkappa_1 \sin \varkappa_2,$$

$$\begin{vmatrix} 1,\ 2,\ 7,\ 8 \\ a,\ b,\ f,\ g \end{vmatrix} = - \Phi_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2,$$

$$\begin{vmatrix} 1,\ 2,\ 7,\ 8 \\ a,\ b,\ f,\ h \end{vmatrix} = - \Phi_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Sin}\, \varkappa_1 \sin \varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&b,&g,&h\end{vmatrix} = +\varkappa_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)^2,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&c,&e,&f\end{vmatrix} = +\Phi_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Cof}\,\varkappa_1\cos\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&c,&e,&g\end{vmatrix} = 0,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&c,&e,&h\end{vmatrix} = -\Phi_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Cof}^2\varkappa_1,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&c,&f,&g\end{vmatrix} = +\Phi_1\varkappa_1(\varkappa_1^2+\varkappa_2^2)\cos^2\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&c,&f,&h\end{vmatrix} = (\varkappa_1^2+\varkappa_2^2)[-\varkappa_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)+\Phi_1(\varkappa_1\sin\varkappa_2\cos\varkappa_2-\varkappa_2\,\mathfrak{Sin}\,\varkappa_1\,\mathfrak{Cof}\,\varkappa_1)],$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&c,&g,&h\end{vmatrix} = +\Phi_1\varkappa_1(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Cof}\,\varkappa_1\cos\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&d,&e,&f\end{vmatrix} = +\Phi_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Cof}\,\varkappa_1\sin\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&d,&e,&g\end{vmatrix} = +\Phi_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Cof}^2\varkappa_1,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&d,&e,&h\end{vmatrix} = 0,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&d,&f,&g\end{vmatrix} = (\varkappa_1^2+\varkappa_2^2)[+\varkappa_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)+\Phi_1(\varkappa_1\sin\varkappa_2\cos\varkappa_2+\varkappa_2\,\mathfrak{Sin}\,\varkappa_1\,\mathfrak{Cof}\,\varkappa_1)],$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&d,&f,&h\end{vmatrix} = +\Phi_1\varkappa_1(\varkappa_1^2+\varkappa_2^2)\sin^2\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\a,&d,&g,&h\end{vmatrix} = +\Phi_1\varkappa_1(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Cof}\,\varkappa_1\sin\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&c,&e,&f\end{vmatrix} = +\Phi_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Sin}\,\varkappa_1\cos\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&c,&e,&g\end{vmatrix} = -\Phi_1\varkappa_1(\varkappa_1^2+\varkappa_2^2)\cos^2\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&c,&e,&h\end{vmatrix} = (\varkappa_1^2+\varkappa_2^2)[+\varkappa_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)-\Phi_1(\varkappa_1\sin\varkappa_2\cos\varkappa_2+\varkappa_2\,\mathfrak{Sin}\,\varkappa_1\,\mathfrak{Cof}\,\varkappa_1)],$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&c,&f,&g\end{vmatrix} = 0,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&c,&f,&h\end{vmatrix} = -\Phi_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Sin}^2\varkappa_1,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&c,&g,&h\end{vmatrix} = +\Phi_1\varkappa_1(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Sin}\,\varkappa_1\cos\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&d,&e,&f\end{vmatrix} = +\Phi_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Sin}\,\varkappa_1\sin\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&d,&e,&g\end{vmatrix} = (\varkappa_1^2+\varkappa_2^2)[-\varkappa_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)+\Phi_1(-\varkappa_1\sin\varkappa_2\cos\varkappa_2+\varkappa_2\,\mathfrak{Sin}\,\varkappa_1\,\mathfrak{Cof}\,\varkappa_1)],$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&d,&e,&h\end{vmatrix} = -\Phi_1\varkappa_1(\varkappa_1^2+\varkappa_2^2)\sin^2\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&d,&f,&g\end{vmatrix} = +\Phi_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Sin}^2\varkappa_1,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&d,&f,&h\end{vmatrix} = 0,$$

$$\begin{vmatrix}1,&2,&7,&8\\b,&d,&g,&h\end{vmatrix} = +\Phi_1\varkappa_1(\varkappa_1^2+\varkappa_2^2)\,\mathfrak{Sin}\,\varkappa_1\sin\varkappa_2,$$

$$\begin{vmatrix}1,&2,&7,&8\\c,&d,&e,&f\end{vmatrix} = +\varkappa_1\varkappa_2(\varkappa_1^2+\varkappa_2^2)^2,$$

$$\left|\begin{matrix}1, 2, 7, 8\\ c, d, e, g\end{matrix}\right| = -\Phi_1 \varkappa_1 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \varkappa_1 \cos \varkappa_2,$$

$$\left|\begin{matrix}1, 2, 7, 8\\ c, d, e, h\end{matrix}\right| = -\Phi_1 \varkappa_1 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \varkappa_1 \sin \varkappa_2,$$

$$\left|\begin{matrix}1, 2, 7, 8\\ c, d, f, g\end{matrix}\right| = -\Phi_1 \varkappa_1 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2,$$

$$\left|\begin{matrix}1, 2, 7, 8\\ c, d, f, h\end{matrix}\right| = -\Phi_1 \varkappa_1 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Sin}\, \varkappa_1 \sin \varkappa_2,$$

$$\left|\begin{matrix}1, 2, 7, 8\\ c, d, g, h\end{matrix}\right| = 0.$$

Substituting these values into the subdeterminants (17) and these again in (16), we obtain, by putting this expression equal zero and by arranging the members, the general buckling condition:

$$(\varkappa_1^2 + \varkappa_2^2)[Z_0 + (\Phi_0 + \Phi_2) Z_1 + (\Psi_0 + \Psi_2) Z_2 + (\Phi_0 \Psi_0 + \Phi_2 \Psi_2) Z_3 +$$
$$+ \Phi_0 \Phi_2 Z_4 + (\Phi_0 \Psi_2 + \Phi_2 \Psi_0) Z_5 + \Psi_0 \Psi_2 Z_6 +$$
$$+ (\Phi_0 \Phi_2 \Psi_0 + \Phi_0 \Phi_2 \Psi_2) Z_7 + (\Phi_0 \Psi_0 \Psi_2 + \Phi_2 \Psi_0 \Psi_2) Z_8 + \Phi_0 \Phi_2 \Psi_0 \Psi_2 Z_9 +$$
$$+ \Phi_1 (Z_{10} + \Phi_0 Z_{11} + \Phi_2 Z_{12} + \Psi_0 Z_{13} + \Psi_2 Z_{14} + \Phi_0 \Psi_0 Z_{15} + \Phi_2 \Psi_2 Z_{16} +$$
$$+ \Phi_0 \Phi_2 Z_{17} + \Phi_0 \Psi_2 Z_{18} + \Phi_2 \Psi_0 Z_{19} + \Psi_0 \Psi_2 Z_{20} + \Phi_0 \Phi_2 \Psi_0 Z_{21} +$$
$$+ \Phi_0 \Phi_2 \Psi_2 Z_{22} + \Phi_0 \Psi_0 \Psi_2 Z_{23} + \Phi_2 \Psi_0 \Psi_2 Z_{24} + \Phi_0 \Phi_2 \Psi_0 \Psi_2 Z_{25})] = 0. \tag{18}$$

In this equation:

$$Z_0 = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, [2\varkappa_1 \varkappa_2 c_1 c_2 e_1 e_2 (\mathfrak{Cof}\, \beta \varkappa_1 \cos \beta \varkappa_2 - 1) + (\varkappa_2^2 c_2^2 e_1^2 - \varkappa_1^2 c_1^2 e_2^2)\, \mathfrak{Sin}\, \beta\, \varkappa_1 \sin \beta \varkappa_2],$$

$$Z_1 = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)^2 (\varkappa_1 c_1 e_2\, \mathfrak{Cof}\, \beta\, \varkappa_1 \sin \beta \varkappa_2 - \varkappa_2 c_2 e_1\, \mathfrak{Sin}\, \beta\, \varkappa_1 \cos \beta \varkappa_2),$$

$$Z_2 = \varkappa_1^2 \varkappa_2^2 (\varkappa_1^2 + \varkappa_2^2)^2 (\varkappa_1 c_1 e_2\, \mathfrak{Sin}\, \beta\, \varkappa_1 \cos \beta \varkappa_2 + \varkappa_2 c_2 e_1\, \mathfrak{Cof}\, \beta\, \varkappa_1 \sin \beta \varkappa_2),$$

$$Z_3 = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)[-\varkappa_1 \varkappa_2 (c_1 e_1 + c_2 e_2) + (\varkappa_1^2 c_1 e_2 - \varkappa_2^2 c_2 e_1)\, \mathfrak{Sin}\, \beta\, \varkappa_1 \sin \beta \varkappa_2 -$$
$$- \varkappa_1 \varkappa_2 (c_2 e_1 + c_1 e_2)\, \mathfrak{Cof}\, \beta\, \varkappa_1 \cos \beta \varkappa_2],$$

$$Z_4 = -\varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)^3\, \mathfrak{Sin}\, \beta\, \varkappa_1 \sin \beta \varkappa_2,$$

$$Z_5 = -\varkappa_1^2 \varkappa_2^2 (\varkappa_1^2 + \varkappa_2^2)^3\, \mathfrak{Cof}\, \beta\, \varkappa_1 \cos \beta \varkappa_2,$$

$$Z_6 = \varkappa_1^3 \varkappa_2^3 (\varkappa_1^2 + \varkappa_2^2)^3\, \mathfrak{Sin}\, \beta\, \varkappa_1 \sin \beta \varkappa_2 = -\varkappa_1^2 \varkappa_2^2 Z_4,$$

$$Z_7 = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)^2 (-\varkappa_1\, \mathfrak{Cof}\, \beta\, \varkappa_1 \sin \beta \varkappa_2 + \varkappa_2\, \mathfrak{Sin}\, \beta\, \varkappa_1 \cos \beta \varkappa_2),$$

$$Z_8 = \varkappa_1^2 \varkappa_2^2 (\varkappa_1^2 + \varkappa_2^2)^2 (-\varkappa_1\, \mathfrak{Sin}\, \beta\, \varkappa_1 \cos \beta \varkappa_2 - \varkappa_2\, \mathfrak{Cof}\, \beta\, \varkappa_1 \sin \beta \varkappa_2),$$

$$Z_9 = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, [2\varkappa_1 \varkappa_2 (\mathfrak{Cof}\, \beta\, \varkappa_1 \cos \beta \varkappa_2 - 1) + (\varkappa_2^2 - \varkappa_1^2)\, \mathfrak{Sin}\, \beta\, \varkappa_1 \sin \beta \varkappa_2],$$

$$Z_{10} = \varkappa_1 \varkappa_2 c_1 c_2 e_1 e_2 (-\varkappa_2\, \mathfrak{Sin}\, \beta\, \varkappa_1 \cos \beta \varkappa_2 + \varkappa_1\, \mathfrak{Cof}\, \beta\, \varkappa_1 \sin \beta \varkappa_2)$$
$$+ \varkappa_1 \varkappa_2 (c_1 e_1 + c_2 e_2)[\varkappa_1 c_1 e_2 (\mathfrak{Cof}\, \varkappa_1 \sin \varkappa_2 + \mathfrak{Cof}\, (\beta - 1)\, \varkappa_1 \sin (\beta - 1)\, \varkappa_2) -$$
$$- \varkappa_2 c_2 e_1 (\mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2 + \mathfrak{Sin}\, (\beta - 1)\, \varkappa_1 \cos (\beta - 1)\, \varkappa_2)]$$
$$+ \varkappa_1^2 c_1^2 e_2^2 [\varkappa_2 \sin \beta \varkappa_2\, \mathfrak{Cof}\, \varkappa_1\, \mathfrak{Cof}\, (\beta - 1)\, \varkappa_1 - \varkappa_1\, \mathfrak{Sin}\, \beta\, \varkappa_1 \sin \varkappa_2 \sin (\beta - 1)\, \varkappa_2]$$
$$+ \varkappa_2^2 c_2^2 e_1^2 [\varkappa_1\, \mathfrak{Sin}\, \beta\, \varkappa_1 \cos \varkappa_2 \cos (\beta - 1)\, \varkappa_2 - \varkappa_2 \sin \beta \varkappa_2\, \mathfrak{Sin}\, \varkappa_1\, \mathfrak{Sin}\, (\beta - 1)\, \varkappa_1],$$

$$Z_{11 \atop 12} = (\varkappa_1^2 + \varkappa_2^2)\Big[-\varkappa_1 \varkappa_2 (c_1 e_1 + c_2 e_2)\, \mathfrak{Sin}\, {}_{(\beta - 1)\, \varkappa_1}^{\varkappa_1} \sin {}_{(\beta - 1)\, \varkappa_2}^{\varkappa_2}$$
$$+ \varkappa_1 c_1 e_2 \Big(\varkappa_1\, \mathfrak{Cof}\, \beta\, \varkappa_1 \sin \varkappa_2 \sin (\beta - 1)\, \varkappa_2 - \varkappa_2 \sin \beta \varkappa_2\, \mathfrak{Sin}\, {}_{(\beta - 1)\, \varkappa_1}^{\varkappa_1}\, \mathfrak{Cof}\, {}_{\varkappa_1}^{(\beta - 1)\, \varkappa_1}\Big)$$
$$+ \varkappa_2 c_2 e_1 \Big(-\varkappa_1\, \mathfrak{Sin}\, \beta\, \varkappa_1 \sin {}_{(\beta - 1)\, \varkappa_2}^{\varkappa_2} \cos {}_{\varkappa_2}^{(\beta - 1)\, \varkappa_2} + \varkappa_2 \cos \beta \varkappa_2\, \mathfrak{Sin}\, \varkappa_1\, \mathfrak{Sin}\, (\beta - 1)\, \varkappa_1\Big)\Big],$$

$$Z_{13 \atop 14} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\Big[-\varkappa_1 \varkappa_2 (c_1 e_1 + c_2 e_2)\, \mathfrak{Cof}\, {}_{(\beta - 1)\, \varkappa_1}^{\varkappa_1} \cos {}_{(\beta - 1)\, \varkappa_2}^{\varkappa_2}$$
$$+ \varkappa_1 c_1 e_2 \Big(\varkappa_1\, \mathfrak{Sin}\, \beta\, \varkappa_1 \cos {}_{(\beta - 1)\, \varkappa_2}^{\varkappa_2} \sin {}_{\varkappa_2}^{(\beta - 1)\, \varkappa_2} - \varkappa_2 \cos \beta \varkappa_2\, \mathfrak{Cof}\, \varkappa_1\, \mathfrak{Cof}\, (\beta - 1)\, \varkappa_1\Big)$$
$$+ \varkappa_2 c_2 e_1 \Big(-\varkappa_1\, \mathfrak{Cof}\, \beta\, \varkappa_1 \cos \varkappa_2 \cos (\beta - 1)\, \varkappa_2 -$$
$$- \varkappa_2 \sin \beta \varkappa_2\, \mathfrak{Cof}\, {}_{(\beta - 1)\, \varkappa_1}^{\varkappa_1}\, \mathfrak{Sin}\, {}_{\varkappa_1}^{(\beta - 1)\, \varkappa_1}\Big)\Big],$$

$$Z_{15 \atop 16} = \varkappa_1 \varkappa_2 (c_1 e_1 + c_2 e_2)\left[\varkappa_2 \mathfrak{Sin}_{(\beta-1)\varkappa_1}^{\varkappa_1} \cos_{(\beta-1)\varkappa_2}^{\varkappa_2} - \varkappa_1 \mathfrak{Cof}_{(\beta-1)\varkappa_1}^{\varkappa_1} \sin_{(\beta-1)\varkappa_2}^{\varkappa_2}\right]$$

$$+ 2\varkappa_1 \varkappa_2\left[\varkappa_1 c_1 e_2 \mathfrak{Cof}_{\varkappa_1}^{(\beta-1)\varkappa_1} \sin_{\varkappa_2}^{(\beta-1)\varkappa_2} - \varkappa_2 c_2 e_1 \mathfrak{Sin}_{\varkappa_1}^{(\beta-1)\varkappa_1} \cos_{\varkappa_2}^{(\beta-1)\varkappa_2}\right]$$

$$+ \varkappa_1^2 c_1 e_2 [\varkappa_1 \mathfrak{Sin}\,\beta\varkappa_1 \sin \varkappa_2 \sin(\beta-1)\varkappa_2 - \varkappa_2 \sin \beta \varkappa_2 \mathfrak{Cof}\,\varkappa_1 \mathfrak{Cof}(\beta-1)\varkappa_1]$$

$$+ \varkappa_1 \varkappa_2\Big[c_1 e_2\Big(-\varkappa_1 \mathfrak{Cof}\,\beta\varkappa_1 \cos_{(\beta-1)\varkappa_2}^{\varkappa_2} \sin_{\varkappa_2}^{(\beta-1)\varkappa_2} +$$

$$+ \varkappa_2 \cos \beta \varkappa_2 \mathfrak{Cof}_{\varkappa_1}^{(\beta-1)\varkappa_1} \mathfrak{Sin}_{(\beta-1)\varkappa_1}^{\varkappa_1}\Big)$$

$$+ c_2 e_1\Big(-\varkappa_1 \mathfrak{Cof}\,\beta\varkappa_1 \sin_{(\beta-1)\varkappa_2}^{\varkappa_2} \cos_{\varkappa_2}^{(\beta-1)\varkappa_2} +$$

$$+ \varkappa_2 \cos \beta \varkappa_2 \mathfrak{Sin}_{\varkappa_1}^{(\beta-1)\varkappa_1} \mathfrak{Cof}_{(\beta-1)\varkappa_1}^{\varkappa_1}\Big)\Big]$$

$$+ \varkappa_2^2 c_2 e_1 [\varkappa_1 \mathfrak{Sin}\,\beta\varkappa_1 \cos \varkappa_2 \cos(\beta-1)\varkappa_2 + \varkappa_2 \sin \beta\varkappa_2 \mathfrak{Sin}\,\varkappa_1 \mathfrak{Sin}(\beta-1)\varkappa_1],$$

$$Z_{17} = (\varkappa_1^2 + \varkappa_2^2)^2 [-\varkappa_1 \mathfrak{Sin}\,\beta\varkappa_1 \sin \varkappa_2 \sin(\beta-1)\varkappa_2 + \varkappa_2 \sin \beta \varkappa_2 \mathfrak{Sin}\,\varkappa_1 \mathfrak{Sin}(\beta-1)\varkappa_1],$$

$$Z_{18 \atop 19} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)^2\Big[-\varkappa_1 \mathfrak{Cof}\,\beta\varkappa_1 \sin_{(\beta-1)\varkappa_2}^{\varkappa_2} \cos_{\varkappa_2}^{(\beta-1)\varkappa_2} +$$

$$+ \varkappa_2 \cos \beta \varkappa_2 \mathfrak{Sin}_{(\beta-1)\varkappa_1}^{\varkappa_1} \mathfrak{Cof}_{\varkappa_1}^{(\beta-1)\varkappa_1}\Big],$$

$$Z_{20} = -\varkappa_1^2 \varkappa_2^2 (\varkappa_1^2 + \varkappa_2^2)^2 [\varkappa_1 \mathfrak{Sin}\,\beta\varkappa_1 \cos \varkappa_2 \cos(\beta-1)\varkappa_2 + \varkappa_2 \sin \beta \varkappa_2 \mathfrak{Cof}\,\varkappa_1 \mathfrak{Cof}(\beta-1)\varkappa_1],$$

$$Z_{21 \atop 22} = (\varkappa_1^2 + \varkappa_2^2)\Big[-2\varkappa_1 \varkappa_2 \mathfrak{Sin}_{\varkappa_1}^{(\beta-1)\varkappa_1} \sin_{\varkappa_2}^{(\beta-1)\varkappa_2}$$

$$+ \varkappa_1\Big(-\varkappa_1 \mathfrak{Cof}\,\beta\varkappa_1 \sin \varkappa_2 \sin(\beta-1)\varkappa_2 + \varkappa_2 \sin \beta \varkappa_2 \mathfrak{Cof}_{(\beta-1)\varkappa_1}^{\varkappa_1} \mathfrak{Sin}_{\varkappa_1}^{(\beta-1)\varkappa_1}\Big)$$

$$+ \varkappa_2\Big(+\varkappa_1 \mathfrak{Sin}\,\beta\varkappa_1 \cos_{(\beta-1)\varkappa_2}^{\varkappa_2} \sin_{\varkappa_2}^{(\beta-1)\varkappa_2} - \varkappa_2 \cos \beta \varkappa_2 \mathfrak{Sin}\,\varkappa_1 \mathfrak{Sin}(\beta-1)\varkappa_1\Big)\Big],$$

$$Z_{23 \atop 24} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\Big[-2\varkappa_1 \varkappa_2 \mathfrak{Cof}_{\varkappa_1}^{(\beta-1)\varkappa_1} \cos_{\varkappa_2}^{(\beta-1)\varkappa_2}$$

$$+ \varkappa_1\Big(-\varkappa_1 \mathfrak{Sin}\,\beta\varkappa_1 \sin_{(\beta-1)\varkappa_2}^{\varkappa_2} \cos_{\varkappa_2}^{(\beta-1)\varkappa_2} + \varkappa_2 \cos \beta \varkappa_2 \mathfrak{Cof}\,\varkappa_1 \mathfrak{Cof}(\beta-1)\varkappa_1\Big)$$

$$+ \varkappa_2\Big(\varkappa_1 \mathfrak{Cof}\,\beta\varkappa_1 \cos \varkappa_2 \cos(\beta-1)\varkappa_2 + \varkappa_2 \sin \beta \varkappa_2 \mathfrak{Sin}_{(\beta-1)\varkappa_1}^{\varkappa_1} \mathfrak{Cof}_{\varkappa_1}^{(\beta-1)\varkappa_1}\Big)\Big]$$

$$Z_{25} = \varkappa_1 \varkappa_2 (\varkappa_1 \mathfrak{Cof}\,\beta\varkappa_1 \sin \beta\varkappa_2 - \varkappa_2 \cos \beta \varkappa_2 \mathfrak{Sin}\,\beta\varkappa_1)$$

$$- 2\varkappa_1^2 \varkappa_2 (\mathfrak{Cof}(\beta-1)\varkappa_1 \sin(\beta-1)\varkappa_2 + \mathfrak{Cof}\,\varkappa_1 \sin \varkappa_2)$$

$$+ 2\varkappa_1 \varkappa_2^2 (\mathfrak{Sin}(\beta-1)\varkappa_1 \cos(\beta-1)\varkappa_2 + \mathfrak{Sin}\,\varkappa_1 \cos \varkappa_2)$$

$$+ \varkappa_1^2 (-\varkappa_1 \mathfrak{Sin}\,\beta\varkappa_1 \sin \varkappa_2 \sin(\beta-1)\varkappa_2 + \varkappa_2 \sin \beta \varkappa_2 \mathfrak{Cof}\,\varkappa_1 \mathfrak{Cof}(\beta-1)\varkappa_1)$$

$$+ \varkappa_2^2 (-\varkappa_1 \mathfrak{Sin}\,\beta\varkappa_1 \cos \varkappa_2 \cos(\beta-1)\varkappa_2 - \varkappa_2 \sin \beta \varkappa_2 \mathfrak{Sin}\,\varkappa_1 \mathfrak{Sin}(\beta-1)\varkappa_1).$$

From this general buckling condition there can be derived a number of special cases concerning the boundary conditions which, however, can only be solved with a great amount of calculation. In the following we shall discuss and evaluate numerically two simple limiting cases, namely, hinged and fixed longitudinal borders, respectively.

b) <u>Hinged longitudinal borders</u>.- For hinged, but rigid mounting $(w = \Delta w = 0)$ of the longitudinal borders $\eta = 0$ and $\eta = b/b_1 = \beta$, we have $T_0 = T_2 = 0$ and $J\eta_0 = J\eta_2 = \infty$, so that

$$\psi_0 = \psi_2 = 0, \qquad \Phi_0 = \Phi_2 = \infty$$

After dividing equation (18) by the factors Φ_0 and Φ_2 the buckling condition becomes:

$$(\kappa_1{}^2 + \kappa_2{}^2) \, [Z_4 + \Phi_1 \, Z_{17}] = 0$$

or

$$\left. \begin{aligned} &- \kappa_1 \kappa_2 \, (\kappa_1{}^2 + \kappa_2{}^2) \, \sinh \beta \kappa_1 \, \sin \beta \kappa_2 \\ &+ \Phi_1 \, [- \kappa_1 \, \sinh \beta \, \kappa_1 \, \sin \kappa_2 \, \sin (\beta-1) \, \kappa_2 + \\ &+ \kappa_2 \, \sin \beta \, \kappa_2 \, \sinh \kappa_1 \, \sinh (\beta-1) \, \kappa_1] = 0 \end{aligned} \right\} \qquad (19)$$

in which the factor $(\kappa_1{}^2 + \kappa_2{}^2)$ has been left out, as it cannot become zero according to hypothesis.

For the limitation, that the ratio of the field widths b_1/b_2 is a real fraction, a further splitting of the buckling condition (19) is possible. This limitation is given by

$$b_1 = r\zeta, \qquad b_2 = s\zeta$$

in which r, s and $r+s = u$ are positive whole numbers. With

$$\frac{\kappa_1}{r} = \kappa_1{}', \qquad \frac{\kappa_2}{r} = \kappa_2{}'$$

the buckling condition (19) becomes

$$\left. \begin{aligned} &- \kappa_1 \kappa_2 (\kappa_1{}^2 + \kappa_2{}^2) \, \sinh u \, \kappa_1{}' \, \sin u \, \kappa_2{}' \\ &\quad + \Phi_2 (- \kappa_1 \sinh u \, \kappa_1{}' \, \sin r \, \kappa_2{}' \, \sin s \, \kappa_2{}' \\ &\quad + \kappa_2 \, \sin u \, \kappa_2 \, \sinh r \kappa_1{}' \, \sinh s \kappa_1{}') = 0 \end{aligned} \right\} \qquad (20)$$

or, since the left side contains the mutual factors $\sinh \kappa_1{}'$ and $\sin \kappa_2{}'$

$$\left. \begin{aligned} \sinh \kappa_1{}' \, \sin \kappa_2{}' \Bigg[&- \kappa_1 \kappa_2 (\kappa_1{}^2 + \kappa_2{}^2) \, \frac{\sinh u \, \kappa_1{}'}{\sinh \kappa_1{}'} \, \frac{\sin u \, \kappa_2{}'}{\sin \kappa_2{}'} \\ &+ \Phi_1 \Bigg(- \kappa_1 \, \frac{\sinh u \, \kappa_1{}'}{\sinh \kappa_1{}'} \, \frac{\sin r \, \kappa_2{}' \, \sin s \, \kappa_2{}'}{\sin \kappa_2{}'} + \\ &+ \kappa_2 \, \frac{\sin u \, \kappa_2{}'}{\sin \kappa_2{}'} \, \frac{\sinh r \, \kappa_1{}' \, \sinh s \, \kappa_1{}'}{\sinh \kappa_1{}'} \Bigg) \Bigg] = 0 \end{aligned} \right\} \qquad (21)$$

in which the quotients in the large brackets are related terms.

The buckling condition (21) is satisfied by putting each factor on the left side equal zero, so that it splits up in two buckling conditions, after dividing through $\sinh \kappa_1'$ (since by hypothesis $\sinh \kappa_1' \neq 0$), which correspond to two basically different buckling forms.

The buckling condition

$$\sin \kappa_2' = 0 \tag{22}$$

is independent of Φ_1; its solutions are

$$\kappa_2 = 0, \quad r\pi, \quad 2r\pi \dots$$

The solution $\kappa_2 = 0$ cancels by hypothesis; to the remaining solutions belong the buckling values

$$k_1 = r^2 \left(\frac{r\pi}{\nu} + \frac{\nu}{r\pi} \right)^2, \quad 4r^2 \left(\frac{2r\pi}{\nu} + \frac{\nu}{2r\pi} \right)^2 \dots$$

and

$$k = r^2 \beta^2 \left(\frac{r\beta\alpha}{m} + \frac{m}{r\beta\alpha} \right)^2, \quad 4r^2\beta^2 \left(\frac{2r\beta\alpha}{m} + \frac{m}{2r\beta\alpha} \right)^2 \dots \tag{23}$$

respectively.

These, however, are the same buckling values for which the nonstiffened plate with the ratio α in the η-direction buckles with $(r + s)$, $2(r + s)$... and in ξ-direction with m half-waves. (See reference 2.)

Along the stiffener there is formed a nodal line, so that the stiffener is twisted but suffers no bending. The dimensions of the stiffener therefore are without influence upon the buckling stresses, according to (22) because of the neglect of its torsional stiffness. These buckling stresses are, in general, so great that, practically, they occur very seldom. (Compare with this the following examples: longitudinal stiffener in the distance $b_1 = \frac{1}{3} b$ and $b_1 = \frac{1}{4} b$, respectively). In most cases, plate and stiffener buckle at the same time at small k values, which are obtained from the buckling condition:

$$(\kappa_1{}^2 + \kappa_2{}^2) - \qquad\qquad\qquad\qquad\qquad\qquad (24)$$

$$\Phi_1 \left(\frac{\sin r \, \kappa_2' \, \sin s \, \kappa_2'}{\kappa_2 \, \sin u \, \kappa_2'} + \frac{\sinh r \, \kappa_1' \, \sinh s \, \kappa_1'}{\kappa_1 \, \sinh u \, \kappa_1'} \right) = 0$$

formed by putting the last factor in equation (21) equal to zero.

Not considering the higher buckling stresses, the buckling values grow according to (24) as the stiffness $\gamma_1 = \gamma$ increases (see following examples) and reach at $\gamma = \infty$; i.e., for a knife-edge mounting of the plate along the stiffener, their maximum values. Since at $\gamma = \infty$ $\Phi_1 = \infty$, the buckling condition for knife-edge mounting along the stiffener is obtained from (24) by placing the factor of Φ_1 equal to zero; the buckling condition is therefore:

$$- \frac{\sin r \, \kappa_2' \, \sin s \, \kappa_2'}{\kappa_2 \, \sin u \, \kappa_2'} + \frac{\sinh r \, \kappa_1' \, \sinh s \, \kappa_1'}{\kappa_1 \, \sinh u \, \kappa_1'} = 0 \qquad (25)$$

c) Examples:

1. Longitudinal stiffener spaced at $b_1 = \frac{1}{3} b$ and $b_1 = \frac{1}{4} b$, respectively.- Having one longitudinal stiffener at $b_1 = \frac{b}{3}$ ($r = 1$, $s = 2$), the buckling stresses are, using (22) and (23), respectively,

$$k = 9 \left(\frac{3\alpha}{m} + \frac{m}{3\alpha} \right)^2, \qquad 36 \left(\frac{6\alpha}{m} + \frac{m}{6\alpha} \right)^2 \ldots$$

In this, the first term (buckling form: 3 half-waves in η- direction (see fig. 3,I) contains the minimum values $k = 9 \times 4 = 36$ with $\frac{3\alpha}{m} = 1$; i.e., for the ratios $\alpha = 1/3$ at $m = 1$, $\alpha = 2/3$ at $m = 2$ ... For a longitudinal stiffener at a distance $b_1 = b/4$, the corresponding buckling values are

$$k = 16 \left(\frac{4\alpha}{m} + \frac{m}{4\alpha} \right)^2 \ldots \geq 64$$

The buckling stresses according to (24) and (25) (buckling forms, see fig. 3,II), are calculated for the same values of γ and δ and plotted against α in figure 4. These

buckling stresses are, even for $\gamma = \infty$, essentially smaller than the ones from (23) with the minimum values k = 36, and 64, respectively, so that the buckling condition (22) has no practical significance.

The buckling-stress curves from (25), i.e., for $\gamma = \infty$ (figs. 4 and 5) have no points of inflection, similar to the axes for $\gamma = \delta = 0$ (nonstiffened plate), but each has only a minimum $\left(k = 10.6 \text{ for } \dfrac{b_1}{b} = \dfrac{1}{3}, \quad k = 8.55 \text{ for } \dfrac{b_1}{b} = \dfrac{1}{4} \right)$, so that the plate buckles in ξ-direction in the sequence 1,2,3 ... half-waves for increasing values of α. However, for finite values of γ some of the curves have points of inflection; for $\gamma = 10$, in the examples, the curve even has a maximum and consequently, two minima. From this fact follows that the plate in the example $b_1/b = 1/3$ and $\gamma = 10$ buckles for increasing values of the ratio α, in the following sequence of the longitudinal waves m:

$\delta = 0$			$\delta = 0.1$		
$0\ \ <\alpha<0.95$		$m = 1$	$0\ \ <\alpha<1.00$		$m = 1$
$0.95<\alpha<1.49$		$m = 2$	$1.00<\alpha<1.26$		$m = 2$
$1.49<\alpha<2.31$		$m = 1$	$1.26<\alpha<2.64$		$m = 1$
$2.31<\alpha<3.02$		$m = 4$			

etc.

Corresponding relations are shown in the example $b_1/b = 1/4$ (fig. 5). For large values of α the curves in their range of validity approach gradually the value of the minimum with the smaller ordinate.

In table 2 the smallest buckling values k are given for the ratios $\alpha = 0.6, 1.0, 1.4, 1.8$. For the accurate calculation of the numerical values the tables of circular and hyperbolic functions by Hayashi (Berlin, 1926) with seven and more figures were used.

TABLE 2

b_1/b	γ	δ	α							
			0.6	m	1.0	m	1.4	m	1.8	m
1/3	5	0	9.849	1	8.959	1	7.852	1	7.756	1
		0.1	9.796	1	8.413	1	7.009	1	6.808	1
		0.2	9.735	1	7.861	1	6.293	1	6.045	1
	10	0	10.215	1	10.697	2	10.244	2	9.754	1
		0.1	10.201	1	10.652	1	9.549	1	8.658	1
		0.2	10.186	1	10.323	1	8.742	1	7.747	1
1/4	5	0	8.062	1	7.015	1	6.643	1	6.997	1
		0.1	8.032	1	6.808	1	6.181	1	6.441	1
		0.2	7.999	1	6.539	1	6.751	1	5.909	1
	10	0	8.362	1	8.266	1	8.096	2	8.182	2
		0.1	8.317	1	8.128	1	7.786	1	7.659	1
		0.2	8.308	1	7.979	1	7.368	1	7.104	1

2. <u>Longitudinal stiffener in the middle of the plate.</u>
If the stiffener lies in the middle of the plate $(b_1 = \frac{b}{2}, \ r = s = 1)$, the plate buckles according to equations (22) and (23) with one nodal line in the middle at the values

$$ k = 4 \left(\frac{2\alpha}{m} + \frac{m}{2\alpha} \right)^2 \ldots \tag{26} $$

the minimum values of which are $k = 16$ at the points $\alpha = \frac{1}{2}$ (m = 1), $\alpha = 1$ (m = 2), etc.

The corresponding buckling forms are antisymmetrical to the center line $\eta = 1$ (fig. 6,I). The buckling condition (24) changes into*

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*The buckling condition (27) is also present in the general solutions by Lokshin (reference 13), in which, for the special case of an arbitrary number of equal stiffeners equally spaced, only the symmetrical buckling forms are considered. While in our case - same as in certain cases for plates with an odd number of longitudinal or transversal stiffeners - the antisymmetrical buckling cases can be derived without difficulties from the symmetrical buckling cases with respect to half the plate width or length, this cannot be done at an even number of stiffeners. But the antisymmetrical buckling cases give in certain regions the smallest buckling values, as will be shown later at examples of a plate with two longitudinal or transversal stiffeners.

$$- 2 \left(\kappa_1{}^2 + \kappa_2{}^2 \right) + \Phi_1 \left(\frac{\tanh \kappa_1}{\kappa_1} - \frac{\tan \kappa_2}{\kappa_2} \right) = 0 \qquad (27)$$

in the limiting case, $\gamma = \infty$, it becomes

$$\frac{\tanh \kappa_1}{\kappa_1} - \frac{\tan \kappa_2}{\kappa_2} = 0 \qquad (28)$$

Equation (28) agrees with the buckling condition of a plate with the width $b/2$, one longitudinal border of which is hinged and the other fixed (reference 8), so that the corresponding buckling form is symmetrical to the center line $\eta = 1$ (fig. 6, IIb).

Timoshenko (reference 2) calculated the smallest buckling stresses for the plate with longitudinal stiffener in the middle by means of the energy method.

Contrary to the example with one longitudinal stiffener at a distance $b_1 = b/3$ or $b_1 = b/4$, in our case, $b_1 = b/2$, the buckling form with the nodal line at the stiffener is of importance. With increasing γ, at first (27) gives the smallest buckling stresses, until at a certain value of γ, which may be denoted as minimum stiffness $(_{min}\gamma)$, the same buckling values as with (26) are obtained. An increase of γ does not lead to a further increase of the buckling stresses, as the smallest buckling stresses are then obtained from the buckling condition (22), and (26), respectively, which is independent of γ, and for which the plate buckles with one nodal line at the stiffener. Hence the value $_{min}\gamma$ is at least required to get the maximum value of a plate with longitudinal stiffener in the middle. The minimum stiffnesses have been calculated numerically by the author in a special paper (reference 19).

3. Dependence of the buckling stresses on the position of the longitudinal stiffener.- The dependence of the buckling stresses k on the ratio b_1/b, is shown in figure 7 for the square plate, for different values of γ (at $\delta = 0$), according to (19). For reasons of symmetry, the curves run symmetrical to the center line $b_1 = 0.5$. In the case of the stiffener lying at the border $\eta = 0$ ($b_1/b = 0$), the buckling values for every value γ, with any but finite magnitude, are equal to those of the plate hinged at

the four sides; thus, $k = 4$ (for one half-wave in longitudinal direction), and $k = 6.25$ (for two half-waves). However, for $\gamma = \infty$, the border $\eta = 0$ is considered as built in, as the buckling stresses in this case take the values $k = 5.74$ and $k = 6.85$, respectively, of a plate fixed at one longitudinal border and hinged at the other points. The buckling values increase for increasing border distance b_1, depending upon the magnitude of γ. For knife-edge mounting along the stiffener ($\gamma = \infty$) the curve (for $m = 1$) reaches its maximum value $k = 25$ for $b_1/b = 0.5$. To this value corresponds a buckling form with one nodal line at the middle stiffener. The corresponding maximum value for $m = 2$ is $k = 16$.

Both curves intersect each other at the point $b_1/b = 0.16$, so that for border distances $\frac{b_1}{b} < 0.16$, we obtain the smallest buckling stresses for one longitudinal wave, and for border distances $\frac{b_1}{b} > 0.16$ for two longitudinal waves. The full line (see fig. 7) consisting of these two curves, which contains the smallest buckling values for $\gamma = \infty$, represents at the same time the upper limit for all buckling stresses occurring in practical cases.

For values $\gamma < \gamma_{min}$ (for instance, $\gamma = 1,3,5$ in fig. 7) plate and stiffener buckle in the entire region with one longitudinal wave. With the value of the minimum stiffness, that is, in this example, $\gamma_{min} = 7.23$, according to table 2 of reference 19, the maximum value $k = 16$ is reached at the point $b_1/b = 0.5$ with $m = 1$. For values $\gamma > \gamma_{min}$ in a mean region, the magnitude of which depends upon γ, the buckling values for $m = 2$ longitudinal waves are determining.

In the neighborhood of the center of the plate the ordinates of the curve for $\gamma = \gamma_{min}$ are practically equal to those of the limiting curve for $\gamma = \infty$ ($m = 2$). In $b_1/b = 0.4$, the difference only amounts to about 2.5 percent. For a stiffener near the middle of the plate the same is true, namely, that an increase of the stiffness over the value γ_{min} does not have as a result an increase of the buckling stresses.

d) <u>Fixed longitudinal borders</u>.— With borders $\eta = 0$ and $\eta = b/b_1 = \beta$ built in, the values

$$J_{\eta 0} = J_{\eta 2} = T_0 = T_2 = \infty$$

and at the same time

$$\Phi_0 = \Phi_2 = \infty, \qquad \psi_0 = \psi_2 = \infty$$

Dividing the general buckling condition (18) through the factors Φ_0, Φ_2, ψ_0, ψ_2 the members containing these factors in the denominator are, with the values above, equal to zero and the buckling condition becomes

$$(\kappa_1^2 + \kappa_2^2)(Z_9 + \Phi_1 Z_{25}) = 0$$

or

$$\left.\begin{aligned}
&\kappa_1 \kappa_2 (\kappa_1^2 + \kappa_2^2) [2\kappa_1\kappa_2(\cosh \beta\kappa_1 \cos \beta\kappa_2 - 1) + \\
&\qquad + (\kappa_2^2 - \kappa_1^2)\sinh \beta\kappa_1 \sin \beta\kappa_2] \\
&+ \Phi_1 \Big\{ \kappa_1\kappa_2 [\kappa_1 \cosh \beta\kappa_1 \sin \beta\kappa_2 - \kappa_2 \cos \beta\kappa_2 \sinh \beta\kappa_1] \\
&- 2\kappa_1^2 \kappa_2 [\cosh(\beta-1)\kappa_1 \sin(\beta-1)\kappa_2 + \cosh \kappa_1 \sin \kappa_2] \\
&+ 2\kappa_1 \kappa_2^2 [\sinh(\beta-1)\kappa_1 \cos(\beta-1)\kappa_2 + \sinh \kappa_1 \cos \kappa_2 \\
&+ \kappa_1^2 [-\kappa_1 \sinh \beta\kappa_1 \sin \kappa_2 \sin(\beta-1)\kappa_2 + \\
&\qquad + \kappa_2 \sin \beta\kappa_2 \cosh \kappa_1 \cosh(\beta-1)\kappa_1] \\
&+ \kappa_2^2 - \kappa_1 \sinh \beta\kappa_1 \cos \kappa_2 \cos(\beta-1)\kappa_2 - \\
&\qquad - \kappa_2 \sin \beta\kappa_2 \sinh \kappa_1 \sinh(\beta-1)\kappa_1] \Big\} = 0
\end{aligned}\right\} \quad (29)$$

In case the stiffener lies in the middle $(\beta = 2)$, the left-hand side of the buckling condition may be reduced to the product of two factors:

$$\left.\begin{aligned}
&\Big\{ \kappa_2 \sinh \kappa_1 \cos \kappa_2 - \kappa_1 \cosh \kappa_1 \sin \kappa_2 \Big\} \times \\
&\times \Big\{ \kappa_1 \kappa_2 (\kappa_1^2 + \kappa_2^2)(\kappa_1 \sinh \kappa_1 \cos \kappa_2 + \kappa_2 \cosh \kappa_1 \sin \kappa_2) \\
&+ \Phi_1 [\kappa_1 \kappa_2 (1 - \cosh \kappa_1 \cos \kappa_2) + \nu^2 \sinh \kappa_1 \sin \kappa_2] \Big\} = 0
\end{aligned}\right\} \quad (30)$$

so that two buckling conditions are formed which are independent of each other.

1. The buckling condition which is formed by putting the first factor in (30) equal to zero:

$$\kappa_2 \sinh \kappa_1 \cos \kappa_2 - \kappa_1 \cosh \kappa_1 \sin \kappa_2 = 0 \qquad (31)$$

is independent of Φ_1, and agrees with the buckling condition for a plate of the width $b/2$, whose one longitudinal border is hinged, the other one being fixed (reference 8). The corresponding buckling form contains therefore a nodal line at the stiffener. The nodal line is identical with hinged mounting of the plate at the same point (antisymmetrical buckling with respect to the axis $\eta = 1$).

2. By placing the second factor in (30) equal to zero, the buckling condition

$$\left. \begin{aligned} &\kappa_1 \kappa_2 (\kappa_1{}^2 + \kappa_2{}^2) (\kappa_1 \sinh \kappa_1 \cos \kappa_2 + \\ &\quad + \kappa_2 \cosh \kappa_1 \sin \kappa_2) + \\ &\quad + \Phi_1 [\kappa_1 \kappa_2 (1 - \cosh \kappa_1 \cos \kappa_2) + \\ &\quad + v^2 \sinh \kappa_1 \sin \kappa_2] = 0 \end{aligned} \right\} \qquad (32)$$

is formed, which gives the buckling stresses of plate and stiffener (symmetrical buckling with respect to the axis $\eta = 1$). To the limiting case $\Phi_1 = 0$ corresponds the nonstiffened plate fixed at the borders $\eta = 0$ and $\eta = 2$; the corresponding buckling condition becomes:

$$\kappa_1 \sinh \kappa_1 \cos \kappa_2 + \kappa_2 \cosh \kappa_1 \sin \kappa_2 = 0 \qquad (33)$$

from equation (32), reference 8.

In figure 8 the buckling values k are plotted against the ratio α. A comparison with the corresponding buckling curves for hinged longitudinal borders shows

(reference 14) that the antisymmetrical buckling form with built-in borders is already determining for considerably smaller values of γ than for hinged longitudinal borders (reference 19).

4. <u>Plate with two longitudinal stiffeners</u>:

a) <u>General buckling condition</u>.- The investigation of the rectangular plate with two longitudinal stiffeners $(r = 3)$ is limited to the case of all four borders hinged.

We have

$$T_0 = T_3 = 0, \quad \gamma_0 = \gamma_3 = \infty$$

and therefore,

$$\psi_0 = \psi_3 = 0, \quad \Phi_0 = \Phi_3 = \infty$$

	Gl.	a A_1	b B_1	c C_1	d D_1	e A_2	f B_2
I	(9″)		1		1		
2	(8″)		$\varkappa_1^2$		$-\varkappa_2^2$		
3	(12′)	$\mathrm{Sin}\,\varkappa_1$	$\mathrm{Cos}\,\varkappa_1$	$\sin\varkappa_2$	$\cos\varkappa_2$	$-\mathrm{Sin}\,\varkappa_1$	$-\mathrm{Cos}\,\varkappa_1$
4	(14′)	$\varkappa_1^2\,\mathrm{Sin}\,\varkappa_1$	$\varkappa_1^2\,\mathrm{Cos}\,\varkappa_1$	$-\varkappa_2^2\sin\varkappa_2$	$-\varkappa_2^2\cos\varkappa_2$	$-\varkappa_1^2\,\mathrm{Sin}\,\varkappa_1$	$-\varkappa_1^2\,\mathrm{Cos}\,\varkappa_1$
5	(13′)	$\varkappa_1\,\mathrm{Cos}\,\varkappa_1$	$\varkappa_1\,\mathrm{Sin}\,\varkappa_1$	$\varkappa_2\cos\varkappa_2$	$-\varkappa_2\sin\varkappa_2$	$-\varkappa_1\,\mathrm{Cos}\,\varkappa_1$	$-\varkappa_1\,\mathrm{Sin}\,\varkappa_1$
6	(15′)	$-\varkappa_1^3\,\mathrm{Cos}\,\varkappa_1 + \Phi_1\,\mathrm{Sin}\,\varkappa_1$	$-\varkappa_1^3\,\mathrm{Sin}\,\varkappa_1 + \Phi_1\,\mathrm{Cos}\,\varkappa_1$	$\varkappa_2^3\cdot\cos\varkappa_2 + \Phi_1\sin\varkappa_2$	$-\varkappa_2^3\sin\varkappa_2 + \Phi_1\cos\varkappa_2$	$+\varkappa_1^3\,\mathrm{Cos}\,\varkappa_1$	$+\varkappa_1^3\,\mathrm{Sin}\,\varkappa_1$
7	(12′)					$\mathrm{Sin}\,\dfrac{b_1+b_2}{b_1}\varkappa_1$	$\mathrm{Cos}\,\dfrac{b_1+b_2}{b_1}\varkappa_1$
8	(14′)					$\varkappa_1^2\,\mathrm{Sin}\,\dfrac{b_1+b_2}{b_1}\varkappa_1$	$\varkappa_1^2\,\mathrm{Cos}\,\dfrac{b_1+b_2}{b_1}\varkappa_1$
9	(13′)					$\varkappa_1\,\mathrm{Cos}\,\dfrac{b_1+b_2}{b_1}\varkappa_1$	$\varkappa_1\,\mathrm{Sin}\,\dfrac{b_1+b_2}{b_1}\varkappa_1$
10	(15′)					$-\varkappa_1^3\,\mathrm{Cos}\,\dfrac{b_1+b_2}{b_1}\varkappa_1 + \Phi_2\,\mathrm{Sin}\,\dfrac{b_1+b_2}{b_1}\varkappa_1$	$-\varkappa_1^3\,\mathrm{Sin}\,\dfrac{b_1+b_2}{b_1}\varkappa_1 + \Phi_2\,\mathrm{Cos}\,\dfrac{b_1+b_2}{b_1}\varkappa_1$
11	(11″)						
12	(10″)						

so that the boundary conditions (8′) to (11′) simplify to

$$Y_1 = 0, \quad (9'') \qquad\qquad Y_3 = 0, \quad (11'')$$
$$Y_1'' = 0, \quad (8'') \qquad\qquad Y_3'' = 0 \quad (10'')$$

The transitional conditions between the fields 1 and 2 are the same as for the plate with one stiffener (table 1); corresponding equations are obtained for stiffener 2. The 4 boundary and 2×4 transitional conditions form, with the solutions:

$$w_{1,2,3} = Y_{1,2,3}\sin\nu\xi, \quad Y_{1,2,3} = A_1\,\mathrm{Sin}\,\varkappa_1\eta + B_1\,\mathrm{Cos}\,\varkappa_1\eta + C_1\sin\varkappa_2\eta + D_1\cos\varkappa_2\eta$$

the denominator determinant, shown in table 3, from which the buckling condition is obtained by putting it equal to zero.

The solution of the determinant is briefly given in the following: After eliminating the lines 1 and 2, the determinant becomes

$$(\varkappa_1^2 + \varkappa_2^2)\begin{vmatrix}1,2\\b,d\end{vmatrix}. \tag{34}$$

The subdeterminant $\begin{vmatrix}1,2\\b,d\end{vmatrix}$ becomes, after elimination of the lines 11 and 12,

$$\begin{vmatrix}1,2\\b,d\end{vmatrix} = (\varkappa_1^2 + \varkappa_2^2)\left(\mathrm{Sin}\,\frac{b}{b_1}\varkappa_1\sin\frac{b}{b_1}\varkappa_2\,\overline{\begin{vmatrix}11,12\\i,l\end{vmatrix}} - \mathrm{Sin}\,\frac{b}{b_1}\varkappa_1\cos\frac{b}{b_1}\varkappa_2\,\overline{\begin{vmatrix}11,12\\i,m\end{vmatrix}} \right.$$
$$\left. - \mathrm{Cos}\,\frac{b}{b_1}\varkappa_1\sin\frac{b}{b_1}\varkappa_2\,\overline{\begin{vmatrix}11,12\\k,l\end{vmatrix}} + \mathrm{Cos}\,\frac{b}{b_1}\varkappa_1\cos\frac{b}{b_1}\varkappa_2\,\overline{\begin{vmatrix}11,12\\k,m\end{vmatrix}}\right). \tag{35}$$

Table 3(cont.)   N.A.C.A. Technical Memorandum No. 904   29

g	h	i	k	l	m
C_2	D_2	A_2	B_2	C_2	D_2
$-\sin \varkappa_2$	$-\cos \varkappa_2$				
$+\varkappa_2^2 \sin \varkappa_2$	$+\varkappa_2^2 \cos \varkappa_2$				
$-\varkappa_2 \cos \varkappa_2$	$+\varkappa_2 \sin \varkappa_2$				
$-\varkappa_2^3 \cos \varkappa_2$	$+\varkappa_2^3 \sin \varkappa_2$				
$\sin \frac{b_1+b_2}{b_1}\varkappa_2$	$\cos \frac{b_1+b_2}{b_1}\varkappa_2$	$-\mathfrak{Sin} \frac{b_1+b_2}{b_1}\varkappa_1$	$-\mathfrak{Cof} \frac{b_1+b_2}{b_1}\varkappa_1$	$-\sin \frac{b_1+b_2}{b_1}\varkappa_2$	$-\cos \frac{b_1+b_2}{b_1}\varkappa_2$
$-\varkappa_2^2 \cdot \sin \frac{b_1+b_2}{b_1}\varkappa_2$	$-\varkappa_2^2 \cos \frac{b_1+b_2}{b_1}\varkappa_2$	$-\varkappa_1^2 \mathfrak{Sin} \frac{b_1+b_2}{b_1}\varkappa_1$	$-\varkappa_1^2 \mathfrak{Cof} \frac{b_1+b_2}{b_1}\varkappa_1$	$+\varkappa_2^2 \cdot \sin \frac{b_1+b_2}{b_1}\varkappa_2$	$+\varkappa_2^2 \cdot \cos \frac{b_1+b_2}{b_1}\varkappa_2$
$\varkappa_2 \cos \frac{b_1+b_2}{b_1}\varkappa_2$	$-\varkappa_2 \sin \frac{b_1+b_2}{b_1}\varkappa_2$	$-\varkappa_1 \mathfrak{Cof} \frac{b_1+b_2}{b_1}\varkappa_1$	$-\varkappa_1 \mathfrak{Sin} \frac{b_1+b_2}{b_1}\varkappa_1$	$-\varkappa_2 \cos \frac{b_1+b_2}{b_1}\varkappa_2$	$+\varkappa_2 \cdot \sin \frac{b_1+b_2}{b_1}\varkappa_2$
$+\varkappa_2^2 \cos \frac{b_1+b_2}{b_1}\varkappa_2$ $+ \Phi_2 \sin \frac{b_1+b_2}{b_1}\varkappa_2$	$-\varkappa_2^2 \cdot \sin \frac{b_1+b_2}{b_1}\varkappa_2$ $+ \Phi_2 \cos \frac{b_1+b_2}{b_1}\varkappa_2$	$+\varkappa_1^2 \mathfrak{Cof} \frac{b_1+b_2}{b_1}\varkappa_1$	$+\varkappa_1^2 \mathfrak{Sin} \frac{b_1+b_2}{b_1}\varkappa_1$	$-\varkappa_2^2 \cdot \cos \frac{b_1+b_2}{b_1}\varkappa_2$	$+\varkappa_2^2 \cdot \sin \frac{b_1+b_2}{b_1}\varkappa_2$
		$\mathfrak{Sin} \frac{b}{b_1}\varkappa_1$	$\mathfrak{Cof} \frac{b}{b_1}\varkappa_1$	$\sin \frac{b}{b_1}\varkappa_2$	$\cos \frac{b}{b_1}\varkappa_2$
		$\varkappa_1^2 \mathfrak{Sin} \frac{b}{b_1}\varkappa_1$	$\varkappa_1^2 \mathfrak{Cof} \frac{b}{b_1}\varkappa_1$	$-\varkappa_2^2 \sin \frac{b}{b_1}\varkappa_2$	$-\varkappa_2^2 \cos \frac{b}{b_1}\varkappa_2$

Since the lines 11 and 12 are independent of the lines 3 to 6, the four 8-line subdeterminants in (35) with $r = i,k$ and $s = l,m$ $(r \neq s)$ may be written, after eliminating first the lines 3 and 4, as

$$\begin{vmatrix} 1, 2, 11, 12 \\ b, d, r, s \end{vmatrix} = (\varkappa_1^2 + \varkappa_2^2)\left[-\mathfrak{Sin}\,\varkappa_1 \sin\varkappa_2 \left(\begin{vmatrix} 3, 4 \\ a, c \end{vmatrix} + \begin{vmatrix} 3, 4 \\ a, g \end{vmatrix} + \begin{vmatrix} 3, 4 \\ c, e \end{vmatrix} - \begin{vmatrix} 3, 4 \\ e, g \end{vmatrix} \right) + \right.$$
$$\left. + \mathfrak{Sin}\,\varkappa_1 \cos\varkappa_2 \left(\begin{vmatrix} 3, 4 \\ a, h \end{vmatrix} - \begin{vmatrix} 3, 4 \\ e, h \end{vmatrix} \right) + \mathfrak{Cof}\,\varkappa_1 \sin\varkappa_2 \left(\begin{vmatrix} 3, 4 \\ c, f \end{vmatrix} - \begin{vmatrix} 3, 4 \\ f, g \end{vmatrix} \right) + \mathfrak{Cof}\,\varkappa_1 \cos\varkappa_2 \begin{vmatrix} 3, 4 \\ f, h \end{vmatrix} \right]. \quad (36)$$

The nine subdeterminants occurring in this equation become, after eliminating the lines 5 and 6,

$$\begin{vmatrix} 3, 4 \\ a, c \end{vmatrix} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2) \left(-\mathfrak{Cof}\,\varkappa_1 \cos\varkappa_2 \begin{vmatrix} 5, 6 \\ e, g \end{vmatrix} - \mathfrak{Cof}\,\varkappa_1 \sin\varkappa_2 \begin{vmatrix} 5, 6 \\ e, h \end{vmatrix} + \right.$$
$$\left. + \mathfrak{Sin}\,\varkappa_1 \cos\varkappa_2 \begin{vmatrix} 5, 6 \\ f, g \end{vmatrix} + \mathfrak{Sin}\,\varkappa_1 \sin\varkappa_2 \begin{vmatrix} 5, 6 \\ f, h \end{vmatrix} \right),$$

$$\begin{vmatrix} 3, 4 \\ a, g \end{vmatrix} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2) \left(\mathfrak{Cof}\,\varkappa_1 \cos\varkappa_2 \begin{vmatrix} 5, 6 \\ e, g \end{vmatrix} - \mathfrak{Sin}\,\varkappa_1 \cos\varkappa_2 \begin{vmatrix} 5, 6 \\ f, g \end{vmatrix} \right) +$$
$$+ \Phi_1 \left(\varkappa_1 \mathfrak{Cof}\,\varkappa_1 \sin\varkappa_2 \begin{vmatrix} 5, 6 \\ e, g \end{vmatrix} - \varkappa_1 \mathfrak{Sin}\,\varkappa_1 \sin\varkappa_2 \begin{vmatrix} 5, 6 \\ f, g \end{vmatrix} - \varkappa_2 \sin^2 \varkappa_2 \begin{vmatrix} 5, 6 \\ g, h \end{vmatrix} \right),$$

$$\begin{vmatrix} 3, 4 \\ c, e \end{vmatrix} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2) \left(\mathfrak{Cof}\,\varkappa_1 \cos\varkappa_2 \begin{vmatrix} 5, 6 \\ e, g \end{vmatrix} + \mathfrak{Cof}\,\varkappa_1 \sin\varkappa_2 \begin{vmatrix} 5, 6 \\ e, h \end{vmatrix} \right) +$$
$$+ \Phi_1 \left(\varkappa_1 \mathfrak{Sin}^2 \varkappa_1 \begin{vmatrix} 5, 6 \\ e, f \end{vmatrix} - \varkappa_2 \mathfrak{Sin}\,\varkappa_1 \cos\varkappa_2 \begin{vmatrix} 5, 6 \\ e, g \end{vmatrix} - \varkappa_2 \mathfrak{Sin}\,\varkappa_1 \sin\varkappa_2 \begin{vmatrix} 5, 6 \\ e, h \end{vmatrix} \right),$$

$$\overline{\begin{vmatrix}3, & 4\\ e, & g\end{vmatrix}} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ e, & g\end{vmatrix}} +$$
$$+ \varPhi_1 \left(- \varkappa_2 \mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ e, & g\end{vmatrix}} + \varkappa_1 \mathfrak{Cof}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ e, & g\end{vmatrix}}\right),$$

$$\overline{\begin{vmatrix}3, & 4\\ a, & h\end{vmatrix}} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2) \left(\mathfrak{Cof}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ e, & h\end{vmatrix}} - \mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & h\end{vmatrix}}\right) +$$
$$+ \varPhi_1 \left(\varkappa_1 \mathfrak{Cof}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ e, & h\end{vmatrix}} - \varkappa_1 \mathfrak{Sin}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & h\end{vmatrix}} + \varkappa_2 \sin \varkappa_2 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ g, & h\end{vmatrix}}\right),$$

$$\overline{\begin{vmatrix}3, & 4\\ e, & h\end{vmatrix}} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ e, & h\end{vmatrix}} +$$
$$+ \varPhi_1 \left(- \varkappa_2 \mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ e, & h\end{vmatrix}} + \varkappa_1 \mathfrak{Cof}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ e, & h\end{vmatrix}}\right),$$

$$\overline{\begin{vmatrix}3, & 4\\ c, & f\end{vmatrix}} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2) \left(\mathfrak{Cof}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & g\end{vmatrix}} + \mathfrak{Cof}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & h\end{vmatrix}}\right) +$$
$$+ \varPhi_1 \left(\varkappa_1 \mathfrak{Sin}\, \varkappa_1 \mathfrak{Cof}\, \varkappa_1 \overline{\begin{vmatrix}5, & 6\\ e, & f\end{vmatrix}} - \varkappa_2 \mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & g\end{vmatrix}} - \varkappa_2 \mathfrak{Sin}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & h\end{vmatrix}}\right),$$

$$\overline{\begin{vmatrix}3, & 4\\ f, & g\end{vmatrix}} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & g\end{vmatrix}} +$$
$$+ \varPhi_1 \left(\varkappa_1 \mathfrak{Cof}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & g\end{vmatrix}} - \varkappa_2 \mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & g\end{vmatrix}}\right),$$

$$\overline{\begin{vmatrix}3, & 4\\ f, & h\end{vmatrix}} = \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \mathfrak{Cof}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & h\end{vmatrix}} +$$
$$+ \varPhi_1 \left(- \varkappa_2 \mathfrak{Sin}\, \varkappa_1 \cos \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & h\end{vmatrix}} + \varkappa_1 \mathfrak{Cof}\, \varkappa_1 \cdot \sin \varkappa_2 \overline{\begin{vmatrix}5, & 6\\ f, & h\end{vmatrix}}\right).$$

Substituting these terms in the subdeterminant (36), we obtain, after arranging the equation:

$$\begin{vmatrix}1, & 2, & 11, & 12\\ b, & d, & r, & s\end{vmatrix} = (\varkappa_1^2 + \varkappa_2^2) \left[\varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2) \overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & f, & h\end{vmatrix}} + \right.$$
$$+ \varPhi_1 \left(- \varkappa_1 \sin^2 \varkappa_2 \overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & f, & g\end{vmatrix}} + \varkappa_2 \mathfrak{Sin}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & g, & h\end{vmatrix}} + \right.$$
$$+ \varkappa_1 \mathfrak{Sin}\, \varkappa_1 \sin \varkappa_2 \overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & e, & f\end{vmatrix}} + \varkappa_2 \mathfrak{Sin}^2 \varkappa_1 \overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & e, & h\end{vmatrix}} +$$
$$\left.\left. + \varkappa_1 \sin \varkappa_2 \cos \varkappa_2 \overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & f, & h\end{vmatrix}} - \varkappa_2 \mathfrak{Sin}\, \varkappa_1 \mathfrak{Cof}\, \varkappa_1 \overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & f, & h\end{vmatrix}}\right)\right]. \tag{37}$$

The calculation of the subdeterminants of fourth order on the right-hand side gives, for the various values of r and s, the following equations:

$r = i, \quad s = l:$

$$\overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & e, & f\end{vmatrix}} = - \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \varPhi_2\, \mathfrak{Cof}\, \frac{b_1 + b_2}{b_1} \varkappa_1 \cos \frac{b_1 + b_2}{b_1} \varkappa_2,$$

$$\overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & e, & h\end{vmatrix}} = + \varkappa_2 (\varkappa_1^2 + \varkappa_2^2)\, \varPhi_2\, \mathfrak{Cof}^2\, \frac{b_1 + b_2}{b_1} \varkappa_1,$$

$$\overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & f, & g\end{vmatrix}} = - \varkappa_1 (\varkappa_1^2 + \varkappa_2^2)\, \varPhi_2 \cos^2 \frac{b_1 + b_2}{b_1} \varkappa_2,$$

$$\overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & f, & h\end{vmatrix}} = (\varkappa_1^2 + \varkappa_2^2) \left[- \varkappa_1 \varkappa_2 (\varkappa_1^2 + \varkappa_2^2) + \right.$$
$$\left. + \varPhi_2 \left(\varkappa_2 \mathfrak{Sin}\, \frac{b_1 + b_2}{b_1} \varkappa_1 \mathfrak{Cof}\, \frac{b_1 + b_2}{b_1} \varkappa_1 - \varkappa_1 \sin \frac{b_1 + b_2}{b_1} \varkappa_2 \cos \frac{b_1 + b_2}{b_1} \varkappa_2\right)\right],$$

$$\overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & g, & h\end{vmatrix}} = - \varkappa_1 (\varkappa_1^2 + \varkappa_2^2)\, \varPhi_2\, \mathfrak{Cof}\, \frac{b_1 + b_2}{b_1} \varkappa_1 \cos \frac{b_1 + b_2}{b_1} \varkappa_2;$$

$r = i, \quad s = m:$

$$\overline{\begin{vmatrix}3, & 4, & 5, & 6\\ a, & c, & e, & f\end{vmatrix}} = - \varkappa_1 (\varkappa_1^2 + \varkappa_2^2)\, \varPhi_2\, \mathfrak{Cof}\, \frac{b_1 + b_2}{b_1} \varkappa_1 \sin \frac{b_1 + b_2}{b_1} \varkappa_2,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,e,\,h\end{matrix}\right| = -\varkappa_2\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\mathfrak{Cof}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\,\sin\frac{b_1+b_2}{b_1}\,\varkappa_2,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,f,\,g\end{matrix}\right| = +(\varkappa_1^2+\varkappa_2^2)\left[\varkappa_1\varkappa_2\,(\varkappa_1^2+\varkappa_2^2)+\right.$$
$$\left.+\,\Phi_2\left(-\varkappa_2\,\mathfrak{Sin}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\,\mathfrak{Cof}\,\frac{b_1+b_2}{b_1}\,\varkappa_1-\varkappa_1\sin\frac{b_1+b_2}{b_1}\,\varkappa_2\cos\frac{b_1+b_2}{b_1}\,\varkappa_2\right)\right],$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,f,\,h\end{matrix}\right| = -\varkappa_1\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\sin^2\frac{b_1+b_2}{b_1}\,\varkappa_2,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,g,\,h\end{matrix}\right| = 0;$$

$$r=k,\quad s=l:$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,e,\,f\end{matrix}\right| = -\varkappa_2\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\mathfrak{Sin}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\cos\frac{b_1+b_2}{b_1}\,\varkappa_2,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,e,\,h\end{matrix}\right| = (\varkappa_1^2+\varkappa_2^2)\left[\varkappa_1\varkappa_2\,(\varkappa_1^2+\varkappa_2^2)+\right.$$
$$\left.+\,\Phi_2\left(\varkappa_2\,\mathfrak{Sin}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\,\mathfrak{Cof}\,\frac{b_1+b_2}{b_1}\,\varkappa_1+\varkappa_1\sin\frac{b_1+b_2}{b_1}\,\varkappa_2\cos\frac{b_1+b_2}{b_1}\,\varkappa_2\right)\right],$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,f,\,g\end{matrix}\right| = 0,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,f,\,h\end{matrix}\right| = \varkappa_2\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\mathfrak{Sin}^2\frac{b_1+b_2}{b_1}\,\varkappa_1,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,g,\,h\end{matrix}\right| = -\varkappa_1\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\mathfrak{Sin}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\cos\frac{b_1+b_2}{b_1}\,\varkappa_2;$$

$$r=k,\quad s=m:$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,e,\,f\end{matrix}\right| = -\varkappa_2\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\mathfrak{Sin}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\sin\frac{b_1+b_2}{b_1}\,\varkappa_2,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,e,\,h\end{matrix}\right| = +\varkappa_1\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\sin^2\frac{b_1+b_2}{b_1}\,\varkappa_2,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,f,\,g\end{matrix}\right| = -\varkappa_2\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\mathfrak{Sin}^2\frac{b_1+b_2}{b_1}\,\varkappa_1,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,f,\,h\end{matrix}\right| = 0,$$

$$\left|\begin{matrix}3,\,4,\,5,\,6\\a,\,c,\,g,\,h\end{matrix}\right| = -\varkappa_1\,(\varkappa_1^2+\varkappa_2^2)\,\Phi_2\,\mathfrak{Sin}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\sin\frac{b_1+b_2}{b_1}\,\varkappa_2.$$

Substituting these subdeterminants in the four terms (37) and those again in (35) and (34), respectively, the following general buckling condition is obtained after multiplying out and arranging the corresponding terms by

$$(\varkappa_1^2+\varkappa_2^2)^4\left(-\varkappa_1^2\varkappa_2^2\,(\varkappa_1^2+\varkappa_2^2)^2\,\mathfrak{Sin}\,\frac{b}{b_1}\,\varkappa_1\sin\frac{b}{b_1}\,\varkappa_2\right.$$
$$+\,\varkappa_1\varkappa_2\,(\varkappa_1^2+\varkappa_2^2)\left[\Phi_1\left(-\varkappa_1\,\mathfrak{Sin}\,\frac{b}{b_1}\,\varkappa_1\sin\varkappa_2\sin\frac{b_2+b_2}{b_1}\,\varkappa_2+\right.\right.$$
$$\left.+\,\varkappa_2\sin\frac{b}{b_1}\,\varkappa_2\,\mathfrak{Sin}\,\varkappa_1\,\mathfrak{Sin}\,\frac{b_2+b_2}{b_1}\,\varkappa_1\right)$$
$$\left.+\,\Phi_2\left(-\varkappa_1\,\mathfrak{Sin}\,\frac{b}{b_1}\,\varkappa_1\sin\frac{b_1+b_2}{b_1}\,\varkappa_2\sin\frac{b_2}{b_1}\,\varkappa_2+\varkappa_2\sin\frac{b}{b_1}\,\varkappa_2\,\mathfrak{Sin}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\,\mathfrak{Sin}\,\frac{b_2}{b_1}\,\varkappa_1\right)\right]$$
$$+\,\Phi_1\,\Phi_2\left\{-\varkappa_1^2\,\mathfrak{Sin}\,\frac{b}{b_1}\,\varkappa_1\sin\varkappa_2\sin\frac{b_2}{b_1}\,\varkappa_2\sin\frac{b_2}{b_1}\,\varkappa_2\right.$$
$$-\,\varkappa_2^2\sin\frac{b}{b_1}\,\varkappa_2\,\mathfrak{Sin}\,\varkappa_1\,\mathfrak{Sin}\,\frac{b_2}{b_1}\,\varkappa_1\,\mathfrak{Sin}\,\frac{b_2}{b_1}\,\varkappa_1$$
$$+\,\varkappa_1\varkappa_2\left[\sin\varkappa_2\,\mathfrak{Sin}\,\frac{b_2}{b_1}\,\varkappa_1\left(\mathfrak{Sin}\,\frac{b_1+b_2}{b_1}\,\varkappa_1\sin\frac{b_2+b_2}{b_1}\,\varkappa_2-\mathfrak{Sin}\,\varkappa_1\sin\frac{b_2}{b_1}\,\varkappa_2\right)\right.$$
$$\left.\left.\left.+\,\mathfrak{Sin}\,\varkappa_1\sin\frac{b_2}{b_1}\,\varkappa_2\left(\sin\frac{b_1+b_2}{b_1}\,\varkappa_2\,\mathfrak{Sin}\,\frac{b_2+b_2}{b_1}\,\varkappa_1-\sin\varkappa_2\,\mathfrak{Sin}\,\frac{b_2}{b_1}\,\varkappa_1\right)\right]\right\}\right) = 0. \tag{38}$$

placing this determinant equal to zero.

b) <u>Two stiffeners, being symmetrical to the center line $\eta = b/2b_1$, with equal dimensions.</u>- For two stiffeners being symmetrical to the middle $\eta = b/2b_1$ with equal cross sections and moments of inertia, Φ_1 becomes

$$\Phi_1 = \Phi_2, \text{ and } b_3 = b_1$$

so that the left-hand side of the buckling condition (38) can be simplified with these values and can be written as a product of two factors. With $b/b_1 = \beta$, the buckling condition becomes

$$
\left.
\begin{aligned}
&\left\{ -\kappa_1\kappa_2 (\kappa_1{}^2 + \kappa_2{}^2) \sinh \beta\kappa_1 \ \sin \beta\kappa_2 \right. \\
&\quad + \Phi_1 [-\kappa_1 \sinh \beta\kappa_1 \ \sin \kappa_2 \sin (\beta - 1)\kappa_2 + \\
&\quad + \kappa_2 \sin \beta \kappa_2 \sinh \kappa_1 \sinh (\beta - 1)\kappa_1] \Big\} \\
&\times \Big\{ \kappa_1\kappa_2 (\kappa_1{}^2 + \kappa_2{}^2) \cosh \beta\kappa_1 \ \cos \beta\kappa_2 \\
&\quad + \Phi_1 [\kappa_1 \cosh \beta \kappa_1 \sin \kappa_2 \cos(\beta - 1)\kappa_2 - \\
&\quad - \kappa_2 \cos \beta \kappa_2 \ \sinh \kappa_1 \cosh(\beta - 1)\kappa_1] \Big\} = 0
\end{aligned}
\right\} \quad (39)
$$

Since each factor, placed equal to zero, satisfies the equation, we get

$$
\left.
\begin{aligned}
1.-\ & -\kappa_1\kappa_2 (\kappa_1{}^2 + \kappa_2{}^2) \sinh \beta\kappa_1 \ \sin \beta\kappa_2 \\
& + \Phi_1 [-\kappa_1 \sinh \beta \kappa_1 \sin \kappa_2 \sin (\beta-1)\kappa_2 + \\
& + \kappa_2 \sin \beta \kappa_2 \sinh \kappa_1 \sinh (\beta-1)\kappa_1] = 0
\end{aligned}
\right\} \quad (40)
$$

This buckling condition agrees with one for a plate of the width $b/2$, hinged at the longitudinal borders, which is stiffened by a longitudinal stiffener at a distance $\eta = 1$. (See buckling condition (19), p. 14.) At the point $\eta = \beta/2$, therefore, the boundary conditions for hinged mounting are satisfied. Thus the buckling form for our case with two stiffeners in the middle of the plate contains a nodal line. The buckling condition (40) therefore gives the buckling stresses for buckling antisymmetrical to the center line $\eta = \beta/2$.

2. The buckling condition:

$$\begin{aligned}
&\kappa_1\kappa_2 \left(\kappa_1{}^2 + \kappa_2{}^2\right) \cosh \beta \kappa_1 \cos \beta \kappa_2 \\
&\quad + \Phi_1 \left[\kappa_1 \cosh \beta \kappa_1 \sin \kappa_2 \cos (\beta - 1) \kappa_2 - \right. \\
&\quad\left. - \kappa_2 \cos \beta \kappa_2 \sinh \kappa_1 \cosh (\beta - 1) \kappa_1\right] = 0
\end{aligned} \tag{41}$$

contains the buckling stresses for buckling symmetrical to the center line $\eta = \beta/2$.

c) Examples:

1. <u>Two longitudinal stiffeners at equal distances.</u>— For two longitudinal stiffeners with equal dimensions and equal distances $(b_1 = b_2 = b_3 = b/3)$ the buckling conditions (40) and (41), after being split up into factors, change into

$$\sinh \frac{\kappa_1}{2} \sin \frac{\kappa_2}{2} \left[- \left(\kappa_1{}^2 + \kappa_2{}^2\right) + \right.$$
$$\left. + \Phi_1 \left(\frac{\sinh \kappa_1}{\kappa_1 \left(3 + 4 \sinh^2 \frac{\kappa_1}{2}\right)} - \frac{\sin \kappa_2}{\kappa_2 \left(3 - 4 \sin^2 \frac{\kappa_2}{2}\right)} \right) \right] = 0 \tag{42}$$

$$\cosh \frac{\kappa_1}{2} \cos \frac{\kappa_2}{2} \left[- \left(\kappa_1{}^2 + \kappa_2{}^2\right) + \right.$$
$$\left. + \Phi_1 \left(\frac{\sinh \kappa_1}{\kappa_1 \left(4 \cosh^2 \frac{\kappa_1}{2} - 3\right)} - \frac{\sin \kappa_2}{\kappa_2 \left(4 \cos^2 \frac{\kappa_2}{2} - 3\right)} \right) \right] = 0 \tag{43}$$

or, if again equating each factor to zero, whereby the factors $\sinh \frac{\kappa_1}{2}$ and $\cosh \frac{\kappa_1}{2}$ are omitted, since they cannot approach zero, according to assumption:

$$\sin \frac{\kappa_2}{2} = 0 \tag{44}$$

$$- \left(\kappa_1{}^2 + \kappa_2{}^2\right) +$$
$$+ \Phi_1 \frac{\sinh \kappa_1}{\kappa_1 \left(3 + 4 \sinh^2 \frac{\kappa_1}{2}\right)} - \frac{\sin \kappa_2}{\kappa_2 \left(3 - 4 \sin^2 \frac{\kappa_2}{2}\right)} = 0 \tag{45}$$

(anti-symmetrical buckling)

$$\cos \frac{\kappa_2}{2} = 0 \qquad\qquad (46)$$

$$-(\kappa_1^2 + \kappa_2^2) +$$

$$+ \Phi_1 \left[\frac{\sinh \kappa_1}{\kappa_1 \left(4 \cosh^2 \frac{\kappa_1}{2} - 3\right)} - \frac{\sin \kappa_2}{\kappa_2 \left(4 \cos^2 \frac{\kappa_2}{2} - 3\right)} \right] = 0 \qquad (47)$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \begin{array}{l}\text{(sym-}\\\text{met-}\\\text{rical}\\\text{buck-}\\\text{ling)}\end{array}$$

To the solutions of (44)

$$\frac{\kappa_2}{2} = \pi, \quad 2\pi, \quad 3\pi \ldots$$

which are independent of Φ_1, correspond the antisymmetrical buckling forms with nodal lines at the longitudinal stiffeners. The buckling stresses corresponding to the first value $\kappa_2 = 2\pi$ are

$$k = 36 \left(\frac{6\alpha}{m} + \frac{m}{6\alpha}\right)^2 \qquad \text{Minima:} \quad k = 144 \quad \text{for} \quad \frac{6\alpha}{m} = 1$$

The buckling form has 6 half-waves in η-direction (fig. 9, Ia). To the following solutions of κ_2 correspond buckling forms with multiples of 6 half-waves. Those values are higher buckling stresses which cannot be taken into account for this calculation.

To the solutions of (46)

$$\frac{\kappa_2}{2} = \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2} \ldots$$

correspond the symmetrical buckling forms with nodal lines at the longitudinal stiffeners. The buckling stresses of the first value are

$$k = 9 \left(\frac{3\alpha}{m} + \frac{m}{3\alpha}\right)^2 \qquad \text{Minima:} \quad k = 36 \quad \text{for} \quad \frac{3\alpha}{m} = 1$$

The buckling form has 3 half-waves in η-direction (fig. 9, IIa). The following solutions of κ_2 also give higher buckling stresses, since to them correspond buckling forms with multiples of 3 half-waves. The equations (45) and (47) give the buckling stresses for the buckling forms sym-

metrical and antisymmetrical to the axis $\eta = \beta/2$, at which the plate buckles with the stiffeners (fig. 9, Ib and IIb).

In figure 10 the buckling stresses for $\gamma = 5$ and 10 are plotted against the ratio α . For small ratios ($\alpha <$ ≈ 0.4) the buckling condition (46) gives the smallest buckling stresses, for great ratios the buckling values have to be calculated from (47). Besides that, for small values of γ (in fig. 10, for $\gamma = 5$) at ratios $\alpha \approx 0.4$ to 0.7 the antisymmetrical buckling is calculated from (45).

Timoshenko calculated approximate values for this case using the energy method and put them together in a table (reference 20), but the table is set up only for the symmetrical buckling cases according to (46) and (47), not for the antisymmetrical buckling form from (45). Several numerical values in the table are therefore too great. For a comparison in figure 10, the curves for $\gamma = 5$ and $\gamma = 10$, respectively, and $\delta = 0.1$, which correspond to Timoshenko's values, are shown as dotted lines. The values of the approximate solution are slightly greater than the exact values from (47); however, with increasing ratio α , the differences become smaller. The solutions by Lokshin (reference 13) are also incomplete as they only consider symmetrical buckling.

2. _Stiffeners at small distances symmetrical to the center line._- With welded structures the case may occur where one longitudinal stiffener is split up in two, each half being on a different side of the plate but both at a small distance from each other (fig. 11), in order to avoid lapping of the welding seams. The buckling stresses can be calculated from (40) and (41) if both halves of the stiffener are symmetrical to the middle. Since b_2 is small against b by hypothesis, the buckling stresses for antisymmetrical buckling (fig. 11,I) increase as b_2 increases; for, in the limiting cases $b_2 = 0$, the plate buckles like the nonstiffened plate with a nodal line, whereas at an eccentricity b_2 also the stiffeners are bent and therefore the resistance against buckling is greater. The buckling stresses therefore remain on the safe side if those of the limiting case $b_2 = 0$ are used. However, the buckling stresses for symmetrical buckling (fig. 11,II) decrease as b_2 increases. For the example of the plate with the ratio $\alpha = 1$, and the stiffener with $\gamma = 2.5$ ($\delta = 0$), the buckling values are, from (41)

b_2/b	0	0.05	0.10	0.15
k	11.06	11.06	11.05	11.00

The differences between the k values are small, so that if no greater eccentricities are in question, the buckling values of the limiting case $b_2 = 0$ can be used.

III. PLATE WITH TRANSVERSAL STIFFENERS

1. _General principles._- The rectangular plate with transversal stiffeners at the points $x = a_1$, $x = a_1 +$ $a_2 \ldots$ (fig. 12) consists of r plate fields with the ratios

$$\alpha_1 = \frac{a_1}{b}, \quad \alpha_2 = \frac{a_2}{b}, \quad \ldots \quad \alpha_r = \frac{a_r}{b}$$

($\alpha = \frac{a}{b} =$ ratio of the whole plate).

With the notations

$$\xi = \frac{x}{b}, \quad \eta = \frac{y}{b}, \quad \sigma_x = - \varphi \, \sigma_e \quad (\sigma_e = \frac{\pi^2 D}{b^2 t} = \text{Euler stress})$$

the deflections w of the center cross section of each plate field i, on account of the compression stresses at the transversal borders $\xi = 0$ and $\xi = \alpha$, satisfy the differential equation (2), in which the value φ_1 is to be replaced by φ, according to the notations mentioned above.

Assuming hinged ($w = \Delta w = 0$) longitudinal borders $\eta = 0$ and $\eta = 1$, the differential equation (2) in each field i is satisfied by

$$w_i = X_i(\xi) \sin n_i \pi \eta \quad (n_i = 1,2,3 \ldots) \qquad (48)$$

by which the bending area in η-direction is assumed to have sine shape. The function X_i, only dependent on ξ, is obtained from the common differential equation

$$X_i^{IV} + (\pi^2 \varphi - 2 n_i^2 \pi^2) X_i'' + n_i^4 \pi^4 X_i = 0 \qquad (49)$$

the general solution of which is

$$X_i = A_i \sin \xi_i \lambda_1 + B_i \cos \xi_i \lambda_1 + C_i \sin \xi_i \lambda_2 + D_i \cos \xi_i \lambda_2 \tag{50}$$

with $\quad \lambda_{\genfrac{}{}{0pt}{}{1}{2}} = \dfrac{\pi}{2} \left(\sqrt{\varphi} \pm \sqrt{\varphi - 4n_i^2} \right) \qquad (\varphi > 4n_i^2)$

Since for every field according to (48) there exists a sine-shaped (in η-direction) bending area, the number of half-waves in η-direction must be the same in all fields ($n_i = n$) because of the steady connection of two adjoining fields. Of all values n the minimum values of the buckling stresses correspond to $n = 1$; therefore, the following general deviations are calculated with $n = 1$. With the aid of the four boundary conditions at the borders $\xi = 0$ and $\xi = \alpha$ and the $4(r - 1)$ transitional conditions at the $(r - 1)$ stiffener we can set up $4r$ homogeneous equations, from which the buckling condition, and from that the buckling stresses $\sigma_k = k\,\sigma_e$ can be calculated.

2. _Boundary and transitional conditions._- At the borders $\xi = 0$ and $\xi = \alpha$, rigid and hinged mounting is assumed; therefore, for $\xi = 0$

$$w_1 = 0 \tag{51}$$

$$\Delta w_1 = 0 \tag{52}$$

and for $\xi = \alpha$

$$w_r = 0 \tag{53}$$

$$\Delta w_r = 0 \tag{54}$$

At each stiffener i, i.e., for $\xi = \alpha_1 + \alpha_2 + \ldots + \alpha_i$ [$i = 1$ to $(r - 1)$], the following transitional conditions must be satisfied:

1. The geometrical conditions for steadiness require

$$w_i = w_{i+1} \tag{55}$$

$$\frac{\partial w_i}{\partial \xi} = \frac{\partial w_{i+1}}{\partial \xi} \tag{56}$$

2. Neglecting the torsional stiffness of the stiffen-
ers, the bending moments m_ξ of adjoining plate fields i
and i + 1 at the stiffener i are equal, i.e.

$$- \frac{D}{b^2} \left(\frac{\partial^2 w_i}{\partial \xi^2} + \mu \frac{\partial^2 w_i}{\partial \eta^2} \right) = - \frac{D}{b^2} \left(\frac{\partial^2 w_{i+1}}{\partial \xi^2} + \mu \frac{\partial^2 w_{i+1}}{\partial \eta^2} \right) \qquad (57a)$$

Since on account of the steady connection the curvatures in
η-direction at the stiffener must be equal, i.e.,

$$\frac{\partial^2 w_i}{\partial \eta^2} = \frac{\partial^2 w_{i+1}}{\partial \eta^2} \qquad (57b)$$

equation (57a) changes into

$$\frac{\partial^2 w_i}{\partial \xi^2} = \frac{\partial^2 w_{i+1}}{\partial \xi^2} \qquad (57)$$

3. Considering the plate fields i and i + 1 as
cut off along the stiffener i, the transversal loading
of the stiffener must be put equal to the difference of
the reaction forces of the two plate fields, at which the
deflections of the stiffener must be assumed to be equal
to the deflections i or i + 1 at the point $\xi = \alpha_1 +$
$\alpha_2 + \dots + \alpha_i$. Denoting the moment of inertia of the stiff-
ener about the ξ-axis by J_i, we have the equation:

$$- E J_i \frac{\partial^4 w_i}{b^4 \partial \eta^4} = - \frac{D}{b^3} \left[\frac{\partial^3 w_i}{\partial \xi^3} + (2-\mu) \frac{\partial^3 w_i}{\partial \xi \partial \eta^2} - \right.$$
$$\left. \frac{\partial^3 w_{i+1}}{\partial \xi^3} - (2-\mu) \frac{\partial^3 w_{i+1}}{\partial \xi \partial \eta^2} \right] \qquad (58a)$$

or with $\frac{E J_i}{b D} = \gamma_i$ and, considering (56) and (57b)

$$\gamma_i \frac{\partial^4 w_i}{\partial \eta^4} = \frac{\partial^3 w_i}{\partial \xi^3} - \frac{\partial^3 w_{i+1}}{\partial \xi^3} \qquad (58)$$

After substituting the solutions

$$w_i = X_i \sin \pi \eta$$

in the boundary and transitional conditions, we obtain the
following equations, only dependent on the variable ξ,

for determining the constants in the functions X_i:

$$\left. \begin{array}{l} X_l = 0 \\ X_l'' = 0 \end{array} \right\} \text{ for } \xi = 0 \qquad \qquad (51')$$
$$\qquad \qquad (52')$$

$$\left. \begin{array}{l} X_r = 0 \\ X_r'' = 0 \end{array} \right\} \text{ for } \xi = \alpha \qquad \qquad (53')$$
$$\qquad \qquad (54')$$

$$\left. \begin{array}{l} X_i - X_{i+1} = 0 \\ X_i' - X_{i+1}' = 0 \\ X_i'' - X_{i+1}'' = 0 \\ - X_i''' + X_{i+1}''' + \gamma_i \pi^4 X_i = 0 \end{array} \right\} \text{ for } \xi = \alpha_1 + \alpha_2 + \ldots + \alpha_i$$

$$(55')$$
$$(56')$$
$$(57')$$
$$(58')$$

3. Plate with one transversal stiffener:

a) Buckling condition.- For the plate with one trans-versal stiffener $(r = 2)$ eight homogeneous equations, ac-cording to $(51')$ to $(58')$, arise from the solutions of fields 1 and 2.

$$w_{\frac{1}{2}} = X_{\frac{1}{2}} \sin \pi \eta, \quad X_{\frac{1}{2}} = A_{\frac{1}{2}} \sin \lambda_1 \xi +$$

$$+ B_{\frac{1}{2}} \cos \lambda_1 \xi + C_{\frac{1}{2}} \sin \lambda_2 \xi + D_{\frac{1}{2}} \cos \lambda_2 \xi$$

the coefficients of which are given in table 4. This de-nominator determinant is solved as for a longitudinal stiffener; therefore we omit the whole process of solution for table 4. The result of the solution gives the buck-ling condition for the uniformly compressed plate with one transversal stiffener:

$$\left. \begin{array}{l} (\lambda_1^2 - \lambda_2^2) \, \lambda_1 \lambda_2 \sin \alpha \lambda_1 \sin \alpha \lambda_2 \\ \quad + \gamma \pi^4 (- \lambda_1 \sin \alpha \lambda_1 \sin \alpha_1 \lambda_1 \sin \alpha_2 \lambda_2 + \\ \quad \quad + \lambda_2 \sin \alpha \lambda_2 \sin \alpha_1 \lambda_1 \sin \alpha_2 \lambda_1) = 0 \end{array} \right\} \qquad (59)$$

The buckling conditions for infinite stiffness γ, i.e., for knife-edge mounting at the stiffener, result from (59) by placing the factor γ equal to 0. The buck-ling condition is

$$- \lambda_1 \sin \alpha \lambda_1 \sin \alpha_1 \lambda_2 \sin \alpha_2 \lambda_2 +$$

$$+ \lambda_2 \sin \alpha \lambda_2 \sin \alpha_1 \lambda_1 \sin \alpha_2 \lambda_1 = 0$$

(60)

Buckling conditions (59) and (60) are, regardless of the position of the stiffener and the magnitude of the stiffness γ, satisfied, if at the same time

$$\sin \alpha \lambda_1 = 0, \qquad \sin \alpha \lambda_2 = 0$$

or

$$\alpha \lambda_1 = m_1 \pi \qquad (m_1 = 0,1,2,3 \ldots)$$

$$\alpha \lambda_2 = m_2 \pi \qquad (m_2 = 0,1,2,3 \ldots)$$

The corresponding ratios α and buckling stresses are obtained from

$$\frac{1}{\alpha} = \frac{\lambda_1}{m_1 \pi} = \frac{\lambda_2}{m_2 \pi}$$

They are

$$\bar{\alpha} = \sqrt{m_1 \, m_2}, \qquad \bar{k} = \frac{(m_1 + m_2)^2}{m_1 \, m_2}.$$

The case m_1 or m_2 equal to zero cancels, as we would get $\bar{\alpha} = 0$ and $\bar{k} = \infty$. With $m_1 = m_2 = 1,2,3 \ldots$ $\bar{\alpha}$ becomes

$$\bar{\alpha} = 1,2,3 \ldots \quad \text{and} \quad \bar{k} = 4$$

This case also has to be excluded since by hypothesis $k > 4$. The values $k = 4$ for the ratios $\alpha = 1,2,3 \ldots$ are the buckling values of the hinged, nonstiffened plate where the buckling form consists of square buckles. It is evident that with one transversal stiffener present, the buckling values, on account of the greater resistance, generally must be greater than the ones of the nonstiffened plate. Only for the special case, where the stiffener coincides with a nodal line of the buckling form of the nonstiffened plate, the solution $m_1 = m_2$ with $k = 4$ is valid also for the stiffened plate.

The other values $\bar{\alpha}$, $\bar{k}$ with $m_1, m_2 = 1,2,3 \ldots$ $(m_1 \neq m_2)$ have a simple significance for the nonstiffened plate.

Table 4

	Gl.	a A_1	b B_1	c C_1	d D_1	e A_2	f B_2	g C_2	h D_2
1	(51')		1		1				
2	(52')		λ_1^2		λ_2^2				
3	(55')	$\sin \alpha_1\,\lambda_1$	$\cos \alpha_1\,\lambda_1$	$\sin \alpha_1\,\lambda_2$	$\cos \alpha_1\,\lambda_2$	$-\sin \alpha_1\,\lambda_1$	$-\cos \alpha_1\,\lambda_1$	$-\sin \alpha_1\,\lambda_2$	$-\cos \alpha_1\,\lambda_2$
4	(57')	$-\lambda_1^2 \sin \alpha_1\,\lambda_1$	$-\lambda_1^2 \cdot \cos \alpha_1\,\lambda_1$	$-\lambda_2^2 \sin \alpha_1\,\lambda_2$	$-\lambda_2^2 \cdot \cos \alpha_1\,\lambda_2$	$+\lambda_1^2 \sin \alpha_1\,\lambda_1$	$+\lambda_1^2 \cos \alpha_1\,\lambda_1$	$+\lambda_1^2 \sin \alpha_1\,\lambda_2$	$+\lambda_1^2 \cdot \cos \alpha_1\,\lambda_2$
5	(56')	$\lambda_1 \cos \alpha_1\,\lambda_1$	$-\lambda_1 \sin \alpha_1\,\lambda_1$	$\lambda_2 \cos \alpha_1\,\lambda_2$	$-\lambda_2 \sin \alpha_1\,\lambda_2$	$-\lambda_1 \cos \alpha_1\,\lambda_1$	$+\lambda_1 \sin \alpha_1\,\lambda_1$	$-\lambda_2 \cos \alpha_1\,\lambda_2$	$+\lambda_2 \sin \alpha_1\,\lambda_2$
6	(58')	$\lambda_1^3 \cos \alpha_1\,\lambda_1$ $+\gamma_1\,\pi^4 \sin \alpha_1\,\lambda_1$	$-\lambda_1^3 \sin \alpha_1\,\lambda_1$ $+\gamma_1\,\pi^4 \cos \alpha_1\,\lambda_1$	$\lambda_2^3 \cos \alpha_1\,\lambda_2$ $+\gamma_1\,\pi^4 \sin \alpha_1\,\lambda_2$	$-\lambda_2^3 \sin \alpha_1\,\lambda_2$ $+\gamma_1\,\pi^4 \cos \alpha_1\,\lambda_2$	$-\lambda_1^3 \cos \alpha_1\,\lambda_1$	$+\lambda_1^3 \sin \alpha_1\,\lambda_1$	$-\lambda_2^3 \cos \alpha_1\,\lambda_2$	$+\lambda_2^3 \sin \alpha_1\,\lambda_2$
7	(53')					$\sin \alpha\,\lambda_1$	$\cos \alpha\,\lambda_1$	$\sin \alpha\,\lambda_2$	$\cos \alpha\,\lambda_2$
8	(54')					$-\lambda_1^2 \sin \alpha\,\lambda_1$	$-\lambda_1^2 \cos \alpha\,\lambda_1$	$-\lambda_2^2 \sin \alpha\,\lambda_2$	$-\lambda_2^2 \cos \alpha\,\lambda_2$

Plotting the buckling values k as functions of the ratio α, the points $\bar{\alpha}, \bar{k}$ agree with the coordinates of the intersections of the buckling curves $k = k(\alpha)$ for the nonstiffened plate which buckles in longitudinal direction with m_1 and m_2 half-waves, as is obvious from equation

$$k = \left(\frac{\alpha}{m_1} + \frac{m_1}{\alpha}\right)^2 = \left(\frac{\alpha}{m_2} + \frac{m_2}{\alpha}\right)^2$$

The result may be summed up in the following sentence: At the points of intersection of the buckling curves for the nonstiffened plate for m_1 and m_2 half-waves in longitudinal direction $(m_1, m_2 = 1,2,3 \ldots; m_1 \neq m_2)$, the buckling condition of the plate, stiffened by one transversal stiffener at any place and with any stiffness γ (also with $\gamma = \infty$) is at the same time satisfied.

For the restriction, ratio a_1/a_2 being a real fraction, we can place

$$a_1 = r \, \zeta, \qquad a_2 = s \, \zeta$$

in which r, s and $u = r + s$ are positive integers.

With the notations

$$\frac{\lambda_1 \zeta}{b} = \lambda_1{}', \qquad \frac{\lambda_2 \zeta}{b} = \lambda_2{}'$$

the buckling condition (59) becomes

$$\left.\begin{aligned}
(\lambda_1{}^2 - \lambda_2{}^2) \, \lambda_1 \, \lambda_2 \sin u \, \lambda_1{}' \sin u \, \lambda_2{}' \\
+ \gamma \, \pi^4 (- \lambda_1 \sin u \, \lambda_1{}' \sin r \, \lambda_2{}' \sin s \, \lambda_2{}' + \\
+ \lambda_2 \sin u \, \lambda_2{}' \sin r \lambda_1{}' \sin s \, \lambda_1{}') = 0
\end{aligned}\right\} \qquad (61)$$

Each term on the left-hand side contains the common factors $\sin \lambda_1{}'$ and $\sin \lambda_2{}'$, so that (61) may be split up into

$$\sin \lambda_1{}' = 0, \qquad \sin \lambda_2{}' = 0 \qquad (62)$$

and

$$
\left.
\begin{aligned}
&(\lambda_1{}^2 - \lambda_2{}^2)\, \lambda_1 \lambda_2 \; \frac{\sin u\, \lambda_1{}'}{\sin \lambda_1{}'} \; \frac{\sin u\, \lambda_2{}'}{\sin \lambda_2{}'} \\[2mm]
&+ \gamma\, \pi^4 \left[- \lambda_1 \frac{\sin u\, \lambda_1{}'}{\sin \lambda_1{}'} \; \frac{\sin r\, \lambda_2{}'\; \sin s\, \lambda_2{}'}{\sin \lambda_2{}'} + \right. \\[2mm]
&\qquad\quad \left. + \lambda_2 \frac{\sin u\, \lambda_2{}'}{\sin \lambda_2{}'} \; \frac{\sin r\, \lambda_1{}'\; \sin s\, \lambda_1{}'}{\sin \lambda_1{}'} \right] = 0
\end{aligned}
\right\} \quad (63)
$$

The quotients in the last equation are related terms.

The buckling conditions (62) which are independent of γ, are satisfied for $\lambda_1{}' = \lambda_2{}' = \pi,\ 2\pi,\ 3\pi,\dots\ m\,\pi\ \dots$ The corresponding buckling values are, both for $\lambda_1{}'$ and $\lambda_2{}'$

$$
k = \left(\frac{bm}{\zeta} + \frac{\zeta}{bm} \right)^2 \qquad (m = 1,2,3\ \dots)
$$

Since $\zeta = \dfrac{a_1}{r} = \dfrac{a_2}{s}$, the fields 1 and 2 buckle like plates hinged at the four borders with mr and ms half-waves in longitudinal direction, so that the entire plate buckles with $m(r + s)$ half-waves and has a nodal line at the stiffener. According to buckling condition (63) plate and stiffener buckle at the same time.

b) Examples:

1. Transversal stiffener at a distance $a_1 = a/3$ $(a_1 = a/4)$.

α) <u>Buckling stresses (including the higher ones) for the square plate</u>.- For the example of the square plate with one transversal stiffener at one-third of the length a the buckling stresses from (62) and (63) are plotted against γ in figure 13. In the buckling conditions we have to place

$$
r = 1, \qquad s = 2, \qquad \frac{\zeta}{b} = \frac{1}{3}
$$

The points of intersection of the curves (63) with the k-axis are at the same time the buckling stresses of the nonstiffened plate $(\gamma = 0)$; the corresponding buckling values are:

k = 4   (buckling form: 1 half-wave in longitudinal
                    direction)

k = 6.25   (buckling form: 2 half-waves in longitudinal
                    direction)

k = 18.06   (buckling form: 4 half-waves in longitudinal
                    direction)

With increasing γ, the buckling stress $k = 4$ ($\gamma =$ 0) goes up first and approaches then asymptotically the value $k = 5.795$, to which corresponds accordingly a stiffener with infinite stiffness. With $k = 5.795$, we have reached the greatest, in practical cases occurring, buckling value; the curves above contain higher buckling stresses. To the values $5.795 < k < 6.5$ correspond negative values γ. These buckling stresses are without physical significance since there are only positive ratios between bending stiffnesses of the stiffener and plate. From (64) we obtain the further solutions:

$$\lambda_1' = \lambda_2' = \pi, \ 2\pi \ldots \quad \text{with} \quad k = \left(3m + \frac{1}{3m}\right)^2 \ (m = 1,2\ldots)$$

The minimum value occurs for $m = 1$ and is $k = \frac{100}{9} =$ 11.11 .... At this value the plate buckles with 3 half-waves in ξ-direction with a node at the stiffener.

β) $\underline{\text{Buckling stresses depending on the ratio}} \ \alpha.-$ In figure 14 the buckling values are plotted as functions of α for the case of one transversal stiffener at $a/3$. The buckling conditions (62) and (63) are for this case

$$\sin \frac{\alpha}{3} \lambda_1 = \sin \frac{\alpha}{3} \lambda_2 = 0 \tag{64}$$

$$\left.\begin{array}{l} (\lambda_1^2 - \lambda_2^2)\lambda_1 \lambda_2 \left(4 \cos^2 \frac{\alpha}{3} \lambda_1 - 1\right)\left(4 \cos^2 \frac{\alpha}{3} \lambda_2 - 1\right) \\[2mm] + \gamma \pi^4 \left[- \lambda_1 \left(4 \cos^2 \frac{\alpha}{3} \lambda_1 - 1\right) \sin \frac{2\alpha}{3} \lambda_2 + \right. \\[2mm] \left. + \lambda_2 \left(4 \cos^2 \frac{\alpha}{3} \lambda_2 - 1\right) \sin \frac{2\alpha}{3} \lambda_1\right] = 0 \end{array}\right\} \tag{65}$$

The buckling values

$$k = \left(\frac{3m}{\alpha} + \frac{\alpha}{3m}\right)^2 \qquad (m = 1,2 \ldots)$$

from (64), which are independent of γ, are those of the nonstiffened plate with 3, 6, 9 ... half-waves in longitudinal direction, so that a nodal line occurs at the stiffener.

The solutions of the equation (65) give curves that are calculated for $\gamma = 0.5$, 1.0, and , and plotted in figure 14. These curves coming from infinity with $\alpha = 0$ have a wave-shaped course and approach - not mentioning those for higher buckling stresses - with increasing ratio α asymptotically the minimum value of the buckling stresses $k = 4$.

For the ratios $\alpha = 3$, 6, 9 ... and those in the immediate neighborhood, the buckling condition (64) gives the smallest buckling stresses.

The initial and end points of the sections, within which according to (64) lie the smaller buckling values, are given by the intersections of the curves for (64) with those for (65), the magnitude of these sections being dependent on the magnitude of the stiffness γ. The sections are smallest for $\gamma = 0$; the corresponding ratios α then lie within the limits

$$\sqrt{(3m - 1)\, 3m} < \alpha < \sqrt{3m\,(3m + 1)}$$

(For $m = 1$, α is $2.450 < \alpha < 3.464$.) The sections become larger with increasing γ and reach their maxima for $\gamma = \infty$. (For $m = 1$, we get $2.12 < \alpha < 3.68$.)

Buckling condition (65) is satisfied independently of γ, if at the same time

$$4 \cos^2 \frac{\alpha}{3} \lambda_1 - 1 = 0 \quad \text{and} \quad 4 \cos^2 \frac{\alpha}{3} \lambda_2 - 1 = 0$$

or

$$\cos \frac{\alpha}{3} \lambda_1 = \pm \frac{1}{2} \quad \text{and} \quad \cos \frac{\alpha}{3} \lambda_2 = \pm \frac{1}{2}$$

so that

$$\frac{\alpha}{3} \lambda_1 = \frac{\alpha}{3} \lambda_2 = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3} \ldots$$

These values are reached at the intersections of the buckling curves for m_1 and m_2 half-waves of the nonstiffened plate $\left(\overline{\alpha} = \sqrt{m_1 \cdot m_2}, \quad \overline{k} = \frac{(m_1 + m_2)^2}{m_1\, m_2} \right)$ with m_1, $m_2 =$ 1, 2, 4, 5, 7, 8 ... $(m_1 \neq m_2)$.

As figure 14 shows, the curves (65), depending on γ, touch each other at the intersections of the buckling curves for 1 and 2, 2 and 4, 4 and 5, etc. half-waves of the nonstiffened plate, so that in these points the buckling values k are equal to those of the nonstiffened plate for any value of the stiffness γ.

From table 5 the exact buckling values k for the ratios $a_1/a = 1/3$ and $a_1/a = 1/4$ may be taken.

Table 5

a_1/a	γ	α				
		0.4	0.6	1.0	1.4	1.8
1/3	5	11.265	8.703	5.694	4.519	4.317
	10	13.768	10.085	5.748	4.519	4.326
	15	15.840	10.741	5.764	4.519	4.329
1/4	5	10.287	7.474	5.317	4.515	4.458
	10	11.907	8.655	5.402	4.516	4.472
	15	13.210	9.226	5.429	4.516	4.477

2. <u>Transversal stiffener in the middle of the plate.</u> For a transversal stiffener in the middle of the plate $(a_1 = a_2 = a/2)$ the buckling conditions (62) and (63) change into

$$\sin \tfrac{\alpha}{2} \lambda_1 = \sin \tfrac{\alpha}{2} \lambda_2 = 0 \qquad (66)$$

and

$$2(\lambda_1{}^2 - \lambda_2{}^2)\lambda_1 \lambda_2 \cos \tfrac{\alpha}{2} \lambda_1 \cos \tfrac{\alpha}{2} \lambda_2 + \gamma \pi^4$$

$$\left[-\lambda_1 \sin \tfrac{\alpha}{2} \lambda_2 \cos \tfrac{\alpha}{2} \lambda_1 + \lambda_2 \sin \tfrac{\alpha}{2} \lambda_1 \cos \tfrac{\alpha}{2} \lambda_2 \right] = 0 \Biggr\} \quad (67)$$

The buckling values from (66)

$$k = \left(\frac{\alpha}{2m} + \frac{2m}{\alpha}\right)^2 \qquad (m = 1, 2, 3 \ldots)$$

are those of the nonstiffened plate with one nodal line in the middle. The corresponding buckling form therefore is antisymmetrical to the center line $\xi = \alpha/2$. Buckling condition (67) which is also contained in Lokshin's solutions (reference 13) gives the buckling values for the buckling form which is symmetrical to the center line $\xi = \alpha/2$. The required minimum stiffnesses were calculated by Timoshenko (reference 2) with the energy method.

4. Plate with two transversal stiffeners -

a) General buckling condition.- For the plate with two transversal stiffeners the 12 equations in table 6 are developed from the deflections w of the plate field 1 to 3 with the aid of the boundary and transitional conditions (51') to (58'). The solution of the denominator determinant gives the following general buckling condition of the hinged rectangular plate with two transversal stiffeners:

$$
\begin{aligned}
&\lambda_1^2 \lambda_2^2 (\lambda_1^2 - \lambda_2^2)^2 \sin \alpha \lambda_1 \sin \alpha \lambda_2 \\
&+ \pi^4 \lambda_1 \lambda_2 (\lambda_2^2 - \lambda_1^2) \{ \gamma_1 [\lambda_1 \sin \alpha \lambda_1 \sin \alpha_1 \lambda_2 \sin (\alpha_2 + \alpha_3) \lambda_2 \\
&\qquad\qquad - \lambda_2 \sin \alpha \lambda_2 \sin \alpha_1 \lambda_1 \sin (\alpha_2 + \alpha_3) \lambda_1] \\
&\qquad + \gamma_2 [\lambda_1 \sin \alpha \lambda_1 \sin (\alpha_1 + \alpha_2) \lambda_2 \sin \alpha_3 \lambda_2 \\
&\qquad\qquad - \lambda_2 \sin \alpha \lambda_2 \sin (\alpha_1 + \alpha_2) \lambda_1 \cdot \sin \alpha_3 \lambda_1] \} \\
&+ \pi^8 \gamma_1 \gamma_2 \{ \lambda_1^2 \sin \alpha \lambda_1 \sin \alpha_1 \lambda_2 \cdot \sin \alpha_2 \lambda_2 \sin \alpha_3 \lambda_2 \\
&\qquad + \lambda_2^2 \sin \alpha \lambda_2 \sin \alpha_1 \lambda_1 \sin \alpha_2 \lambda_1 \sin \alpha_3 \lambda_1 \\
&\qquad + \lambda_1 \lambda_2 [\sin \alpha_1 \lambda_1 \sin \alpha_3 \lambda_2 (\sin \alpha_1 \lambda_2 \sin \alpha_3 \lambda_1 - \\
&\qquad\qquad - \sin (\alpha_1 + \alpha_2) \lambda_2 \sin (\alpha_2 + \alpha_3) \lambda_1) \\
&\qquad + \sin \alpha_1 \lambda_2 \sin \alpha_3 \lambda_1 (\sin \alpha_1 \lambda_1 \sin \alpha_3 \lambda_2 - \\
&\qquad\qquad - \sin (\alpha_1 + \alpha_2) \lambda_1 \sin (\alpha_2 + \alpha_3) \lambda_2)] \} = 0.
\end{aligned}
\tag{68}
$$

b) Two transversal stiffeners, symmetrical to the center line $\xi = \alpha/2$, with equal dimensions.- The buckling condition (68), with stiffeners placed symmetrically

$$\gamma_1 = \gamma_2 = \gamma \quad \text{and} \quad a_1 = a_3, \quad \alpha_1 = \alpha_3$$

simplifies to

$$
\begin{aligned}
&\left[(\lambda_1^2 - \lambda_2^2) \lambda_1 \lambda_2 \sin \tfrac{\alpha}{2} \lambda_1 \sin \tfrac{\alpha}{2} \lambda_2 + \gamma \pi^4 \left(- \lambda_1 \sin \tfrac{\alpha}{2} \lambda_1 \sin \alpha_1 \lambda_2 \sin \tfrac{\alpha_2}{2} \lambda_2 \right. \right. \\
&\left. + \lambda_2 \sin \tfrac{\alpha}{2} \lambda_2 \sin \alpha_1 \lambda_1 \sin \tfrac{\alpha_2}{2} \lambda_1 \right) \Big] \cdot \Big[(\lambda_1^2 - \lambda_2^2) \lambda_1 \lambda_2 \cos \tfrac{\alpha}{2} \lambda_1 \cos \tfrac{\alpha}{2} \lambda_2 \\
&\left. + \gamma \pi^4 \left(- \lambda_1 \cos \tfrac{\alpha}{2} \lambda_1 \sin \alpha_1 \lambda_2 \cos \tfrac{\alpha_2}{2} \lambda_2 + \lambda_2 \cos \tfrac{\alpha}{2} \lambda_2 \sin \alpha_1 \lambda_1 \cos \tfrac{\alpha_2}{2} \lambda_1 \right) \right] = 0.
\end{aligned}
\tag{69}
$$

From this we obtain two different buckling conditions by placing each of the two factors equal to zero.

1. The first buckling condition

$$
\begin{aligned}
&(\lambda_1^2 - \lambda_2^2) \lambda_1 \lambda_2 \sin \tfrac{\alpha}{2} \lambda_1 \sin \tfrac{\alpha}{2} \lambda_2 \\
&+ \gamma \pi^4 \left(- \lambda_1 \sin \tfrac{\alpha}{2} \lambda_1 \sin \alpha_1 \lambda_2 \sin \tfrac{\alpha_2}{2} \lambda_2 + \lambda_2 \sin \tfrac{\alpha}{2} \lambda_2 \sin \alpha_1 \lambda_1 \sin \tfrac{\alpha_2}{2} \lambda_1 \right) = 0
\end{aligned}
\tag{70}
$$

agrees with the one for the plate with one transversal stiffener at the point $\xi = \alpha_1$ (see equation (59)) where the border of the plate $\xi = \alpha$ in (59) corresponds to the center of the plate $\xi = \alpha/2$ in (70). Since buckling condition (59) is based on hinged plate borders, the buckling form from (70) contains one nodal line in the center of the plate, so that the antisymmetrical buckling forms correspond to buckling condition (70).

2. The second factor in (69) gives the buckling condition

$$
\begin{aligned}
&(\lambda_1^2 - \lambda_2^2) \lambda_1 \lambda_2 \cos \tfrac{\alpha}{2} \lambda_1 \cos \tfrac{\alpha}{2} \lambda_2 \\
&+ \gamma \pi^4 \left(- \lambda_1 \cos \tfrac{\alpha}{2} \lambda_1 \sin \alpha_1 \lambda_2 \cos \tfrac{\alpha_2}{2} \lambda_2 + \lambda_2 \cos \tfrac{\alpha}{2} \lambda_2 \sin \alpha_1 \lambda_1 \cos \tfrac{\alpha_2}{2} \lambda_1 \right) = 0
\end{aligned}
\tag{71}
$$

with the buckling stresses for buckling forms symmetrical to the center line $\xi = \alpha/2$.

		a	b	c	d	e	f
	Gl.	A_1	B_1	C_1	D_1	A_2	B_2
1	(51')		I		I		
2	(52')		λ_1^2		λ_2^2		
3	(55')	$\sin \alpha_1 \lambda_1$	$\cos \alpha_1 \lambda_1$	$\sin \alpha_1 \lambda_2$	$\cos \alpha_1 \lambda_2$	$-\sin \alpha_1 \lambda_1$	$-\cos \alpha_1 \lambda_1$
4	(57')	$-\lambda_1^2 \sin \alpha_1 \lambda_1$	$-\lambda_1^2 \cos \alpha_1 \lambda_1$	$-\lambda_2^2 \sin \alpha_1 \lambda_2$	$-\lambda_2^2 \cos \alpha_1 \lambda_2$	$+\lambda_1^2 \sin \alpha_1 \lambda_1$	$+\lambda_1^2 \cos \alpha_1 \lambda_1$
5	(56')	$\lambda_1 \cos \alpha_1 \lambda_1$	$-\lambda_1 \sin \alpha_1 \lambda_1$	$\lambda_2 \cos \alpha_1 \lambda_2$	$-\lambda_2 \sin \alpha_1 \lambda_2$	$-\lambda_1 \cos \alpha_1 \lambda_1$	$+\lambda_1 \sin \alpha_1 \lambda_1$
6	(58')	$\lambda_1^3 \cos \alpha_1 \lambda_1 + \gamma_1 \pi^4 \sin \alpha_1 \lambda_1$	$-\lambda_1^3 \sin \alpha_1 \lambda_1 + \gamma_1 \pi^4 \cos \alpha_1 \lambda_1$	$\lambda_2^3 \cos \alpha_1 \lambda_2 + \gamma_1 \pi^4 \sin \alpha_1 \lambda_2$	$-\lambda_2^3 \sin \alpha_1 \lambda_2 + \gamma_1 \pi^4 \cos \alpha_1 \lambda_2$	$-\lambda_1^3 \cos \alpha_1 \lambda_1$	$+\lambda_1^3 \sin \alpha_1 \lambda_1$
7	(55')					$\sin (\alpha_1 + \alpha_2) \lambda_1$	$\cos (\alpha_1 + \alpha_2) \lambda_1$
8	(57')					$-\lambda_1^2 \sin (\alpha_1 + \alpha_2) \lambda_1$	$-\lambda_1^2 \cos (\alpha_1 + \alpha_2) \lambda_1$
9	(56')					$\lambda_1 \cos (\alpha_1 + \alpha_2) \lambda_1$	$-\lambda_1 \sin (\alpha_1 + \alpha_2) \lambda_1$
10	(58')					$\lambda_1^3 \cos (\alpha_1 + \alpha_2) \lambda_1 + \gamma_2 \pi^4 \sin (\alpha_1 + \alpha_2) \lambda_1$	$-\lambda_1^3 \sin (\alpha_1 + \alpha_2) \lambda_1 + \gamma_2 \pi^4 \cos (\alpha_1 + \alpha_2) \lambda_1$
11	(53')						
12	(54')						

Condition (70) for antisymmetrical buckling is satisfied for any value γ, if

$$\sin \frac{\alpha}{2} \lambda_1 = \sin \frac{\alpha}{2} \lambda_2 = 0$$

or

$$\frac{\alpha}{2} \lambda_1 = \frac{\alpha}{2} \lambda_2 = \pi, \ 2\pi, \ 3\pi \ldots$$

To these values are coordinated the points with the ratios $\bar{\alpha} = \sqrt{m_1 m_2}$ and the buckling values

$\bar{k} = \dfrac{(m_1 + m_2)^2}{m_1 m_2}$ for $m_1, m_2 = 2, 4, 6 \ldots (m_1 \neq m_2)$; the values with $m_1 = m_2$ drop out for the same reason as in the corresponding case of one transversal stiffener. Buckling condition (71) for symmetrical buckling, independent of position and magnitude of the stiffener, is satisfied for

$$\cos \frac{\alpha}{2} \lambda_1 = \cos \frac{\alpha}{2} \lambda_2 = 0$$

or

$$\frac{\alpha}{2} \lambda_1 = \frac{\alpha}{2} \lambda_2 = \frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{5\pi}{2} \ldots$$

To this correspond the points $\bar{\alpha}$, $\bar{k}$ with $m_1, m_2 = 1, 3, 5 \ldots (m_1 \neq m_2)$.

c) Examples.— For two equally large stiffeners at equal distances $(a_1 = a_2 = a/3)$ the buckling conditions (70) and (71) become, after further splitting up:

$$\sin \frac{\alpha}{6} \lambda_1 = \sin \frac{\alpha}{6} \lambda_2 = 0, \tag{72}$$

$$\left. \begin{array}{l} (\lambda_1^2 - \lambda_2^2) \lambda_1 \lambda_2 \left(4 \cos^2 \frac{\alpha}{6} \lambda_1 - 1 \right) \left(4 \cos^2 \frac{\alpha}{6} \lambda_2 - 1 \right) \\[2mm] \quad + \gamma \pi^4 \left[-\lambda_1 \left(4 \cos^2 \frac{\alpha}{6} \lambda_1 - 1 \right) \sin \frac{\alpha}{3} \lambda_2 + \lambda_2 \left(4 \cos^2 \frac{\alpha}{6} \lambda_2 - 1 \right) \sin \frac{\alpha}{3} \lambda_1 \right] = 0 \end{array} \right\} \tag{73}$$

g	h	i	k	l	m
C_2	D_2	A_3	B_3	C_3	D_3
$-\sin \alpha_1 \lambda_2$	$-\cos \alpha_1 \lambda_2$				
$+\lambda_2^2 \sin \alpha_1 \lambda_2$	$+\lambda_2^2 \cos \alpha_1 \lambda_2$				
$-\lambda_2 \cos \alpha_1 \lambda_2$	$+\lambda_2 \sin \alpha_1 \lambda_2$				
$-\lambda_2^3 \cos \alpha_1 \lambda_2$	$+\lambda_2^3 \sin \alpha_1 \lambda_2$				
$\sin (\alpha_1 + \alpha_2) \lambda_2$	$\cos (\alpha_1 + \alpha_2) \lambda_2$	$-\sin (\alpha_1 + \alpha_2) \lambda_1$	$-\cos (\alpha_1 + \alpha_2) \lambda_1$	$-\sin (\alpha_1 + \alpha_2) \lambda_2$	$-\cos (\alpha_1 + \alpha_2) \lambda_2$
$-\lambda_2^2 \sin (\alpha_1 + \alpha_2) \lambda_2$	$-\lambda_2^2 \cos (\alpha_1 + \alpha_2) \lambda_2$	$+\lambda_1^2 \sin (\alpha_1 + \alpha_2) \lambda_1$	$+\lambda_1^2 \cos (\alpha_1 + \alpha_2) \lambda_1$	$+\lambda_2^2 \sin (\alpha_1 + \alpha_2) \lambda_2$	$+\lambda_2^2 \cos (\alpha_1 + \alpha_2) \lambda_2$
$\lambda_2 \cos (\alpha_1 + \alpha_2) \lambda_2$	$-\lambda_2 \sin (\alpha_1 + \alpha_2) \lambda_2$	$-\lambda_1 \cos (\alpha_1 + \alpha_2) \lambda_1$	$+\lambda_1 \sin (\alpha_1 + \alpha_2) \lambda_1$	$-\lambda_2 \cos (\alpha_1 + \alpha_2) \lambda_2$	$+\lambda_2 \sin (\alpha_1 + \alpha_2) \lambda_2$
$\lambda_2^3 \cos (\alpha_1 + \alpha_2) \lambda_2$ $+\gamma_2 \pi^4 \sin (\alpha_1 + \alpha_2) \lambda_2$	$-\lambda_2^3 \sin (\alpha_1 + \alpha_2) \lambda_2$ $+\gamma_2 \pi^4 \cos (\alpha_1 + \alpha_2) \lambda_2$	$-\lambda_1^3 \cos (\alpha_1 + \alpha_2) \lambda_1$	$+\lambda_1^3 \sin (\alpha_1 + \alpha_2) \lambda_1$	$-\lambda_2^3 \cos (\alpha_1 + \alpha_2) \lambda_2$	$+\lambda_2^3 \sin (\alpha_1 + \alpha_2) \lambda_2$
		$\sin \alpha \lambda_1$	$\cos \alpha \lambda_1$	$\sin \alpha \lambda_2$	$\cos \alpha \lambda_2$
		$\lambda_1^2 \sin \alpha \lambda_1$	$\lambda_1^2 \cos \alpha \lambda_1$	$\lambda_2^2 \sin \alpha \lambda_2$	$\lambda_2^2 \cos \alpha \lambda_2$

for antisymmetrical buckling, and

$$\cos \frac{\alpha}{6} \lambda_1 = \cos \frac{\alpha}{6} \lambda_2 = 0, \tag{74}$$

$$(\lambda_1^2 - \lambda_2^2) \lambda_1 \lambda_2 \left(4 \cos^2 \frac{\alpha}{6} \lambda_1 - 3\right)\left(4 \cos^2 \frac{\alpha}{6} \lambda_2 - 3\right)$$
$$+ \gamma \pi^4 \left[- \lambda_1 \left(4 \cos^2 \frac{\alpha}{6} \lambda_1 - 3\right) \sin \frac{\alpha}{3} \lambda_2 + \lambda_2 \left(4 \cos^2 \frac{\alpha}{6} \lambda_2 - 3\right) \sin \frac{\alpha}{3} \lambda_1\right] = 0 \tag{75}$$

for symmetrical buckling.

    The corresponding buckling stresses are given in figure 15 for different values.

    The equations for antisymmetrical buckling agree with those for a plate with one transversal stiffener at 1/3 of its length. (See (64) and (65), also fig. 14.) The ratios α in those equations must be replaced by $\alpha/2$. The discussion of antisymmetrical buckling is omitted since the buckling conditions (64) and (65) have been discussed before.

    From (74) we obtain the buckling values:

$$k = \left(\frac{3m}{\alpha} + \frac{\alpha}{3m}\right)^2 \quad (m = 1, 2, 3 \dots).$$

    The plate buckles like the nonstiffened plate with 3, 6, 9 ... half-waves in longitudinal direction, so that nodal lines arise at the stiffeners. The buckling stresses from (75) give the curves for the symmetrical buckling forms, for which the stiffeners also bend at buckling. These curves, like the corresponding curves of the antisymmetrical buckling, also approach asymptotically the minimum value $k = 4$ for increasing ratio α.

Since buckling condition (75) is satisfied for any value γ, if

$$4 \cos^2 \frac{\alpha}{6} \lambda_1 - 3 = 4 \cos^2 \frac{\alpha}{6} \lambda_2 - 3 = 0$$

or

$$\frac{\alpha}{6} \lambda_1 = \frac{\alpha}{6} \lambda_2 = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \ldots$$

it follows for the corresponding values

$$\alpha = \sqrt{m_1 \, m_2} \quad \text{and} \quad \bar{k} = \frac{(m_1 + m_2)^2}{m_1 \, m_2}$$

that $m_1, m_2 = 1, 5, 7 \ldots$ Therefore the curves, depending on γ, touch each other at the intersections of the curves for 1 and 5, 5 and 7 .. half-waves of the nonstiffened plate. At the intersections of the curves for 1 and 3, 3 and 5 ... half-waves of the nonstiffened plate, the buckling condition (74), which is independent of γ, confirms the already mentioned fact, that also in these points the plate buckles independently of the stiffeners.

In the sections in which the buckling conditions give the smallest buckling values, the curves are full lines. It is evident from the continuous change of symmetrical and antisymmetrical buckling forms that both types are of equal significance.

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         III.

Figure 1.- Notations of the plate with longitudinal stiffeners.
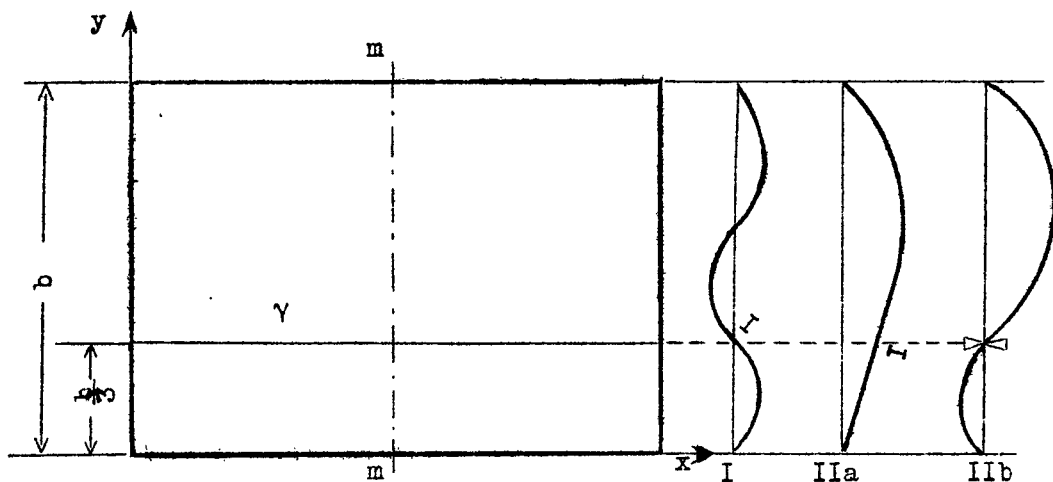
Figure 3.- Cross-sections through the possible buckling forms
  I, Nodal line at the stiffener
  IIa,Buckling of plate and stiffener
  IIb,Knife edge mounting of stiffener
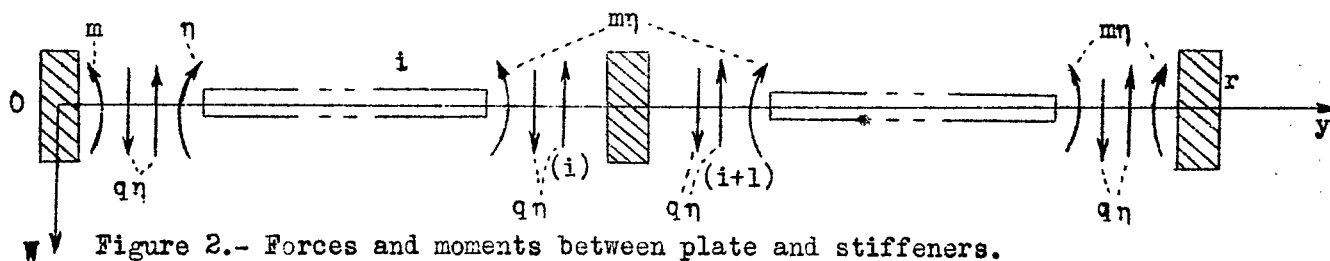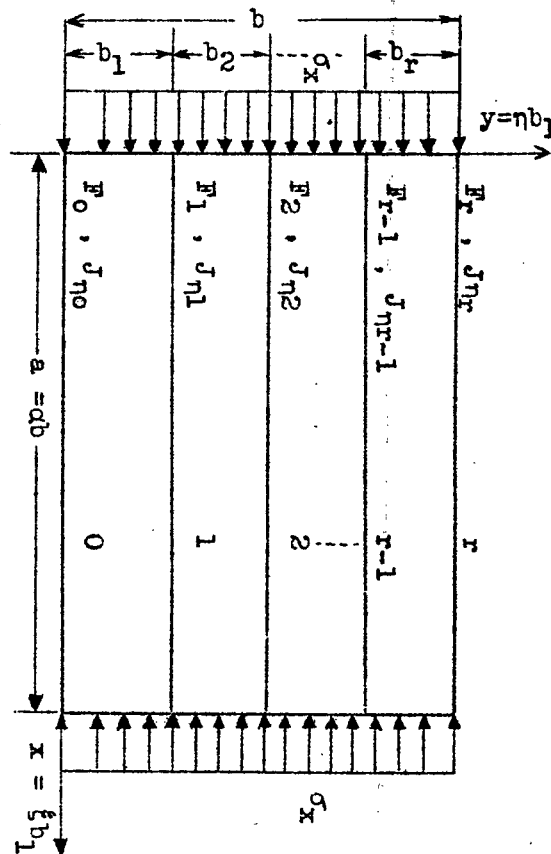
Figure 2.- Forces and moments between plate and stiffeners.

Figure 4.- Buckling stresses of a plate with one longitudinal stiffener at 1/3 of the width b.
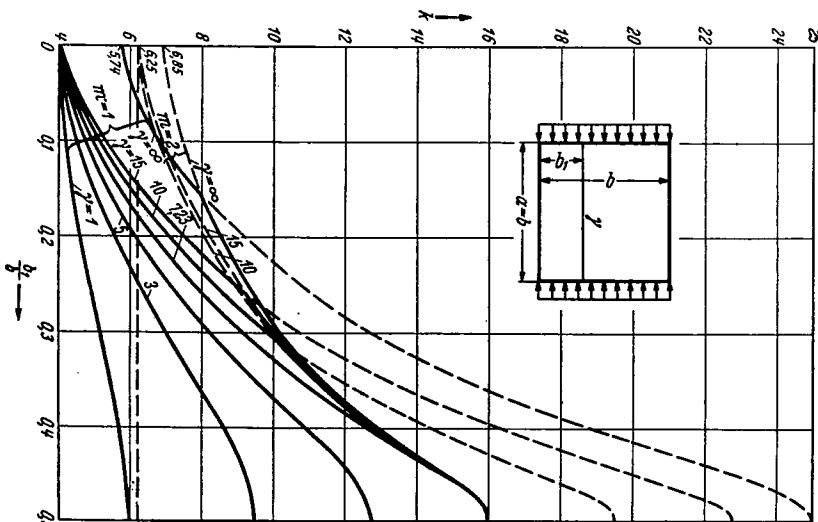
Figure 7.- Buckling stresses as function of the position of the stiffener. (Square plate with one longitudinal stiffener). (Hinged longitudinal borders).
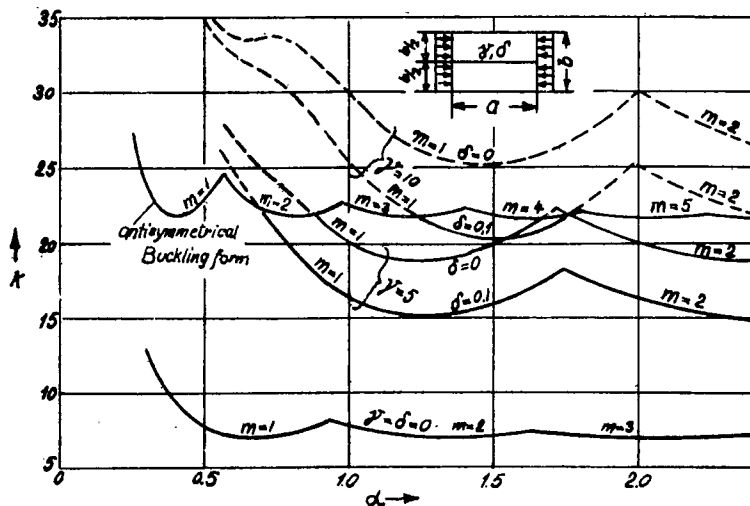
Figure 8.- Buckling stresses of a plate with one longitudinal stiffener in the middle. (Built-in longitudinal borders)
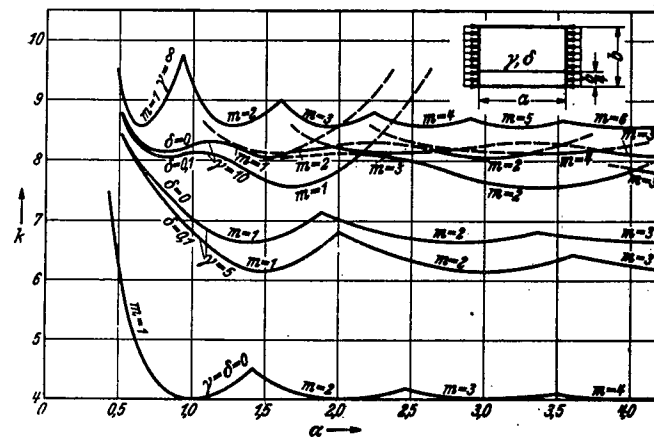
Figure 5.- Buckling stresses of a plate with one longitudinal stiffener at 1/4 of the width b.
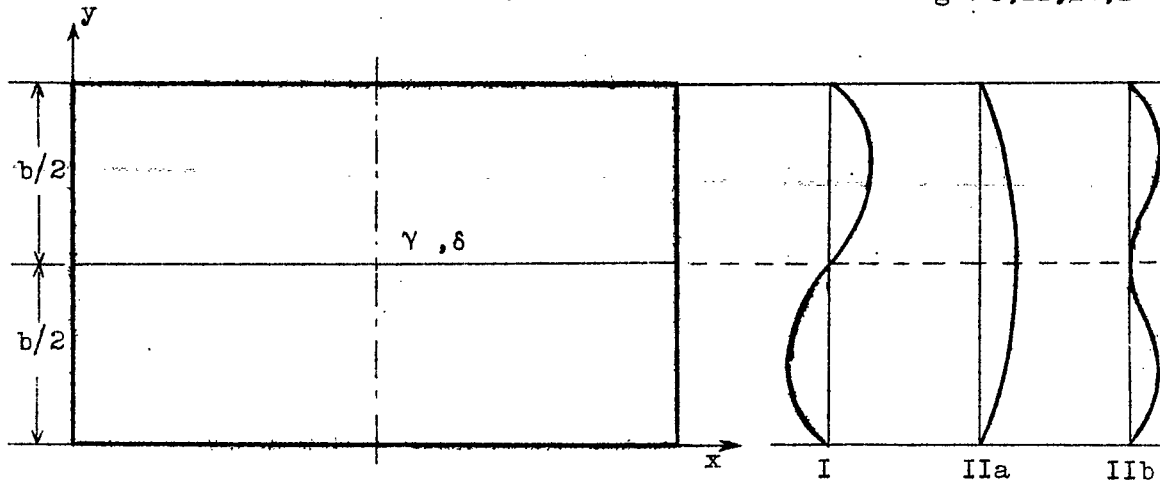
Figure 6.- Buckling forms of a plate with longitudinal stiffener in the middle. I, antisymmetrical buckling form; II, symmetrical buckling forms; (a) γ = finite value, (b) $\gamma = \infty$, (knife-edge mounting).
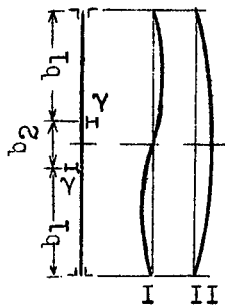
Figure 11
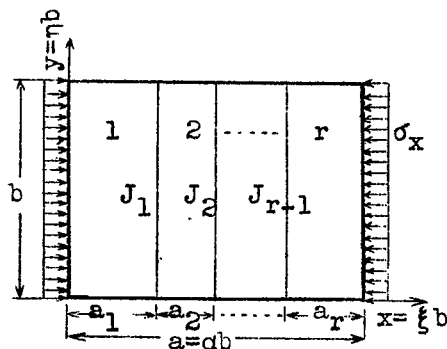
Figure 12.- Notations of plate with transversal stiffeners.
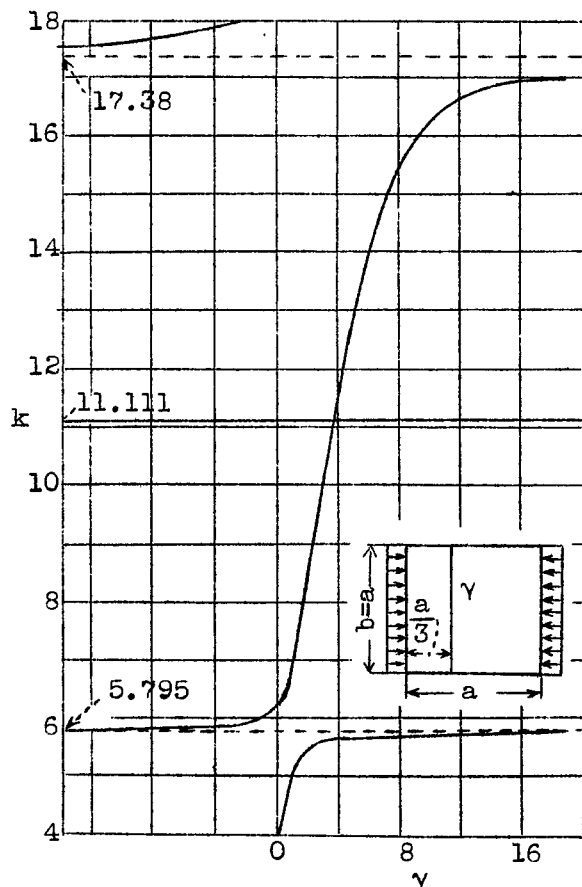
Figure 13.- Buckling stresses vs. stiffness of transversal stiffener. (Square plate with one stiffener at 1/3 of length a.)
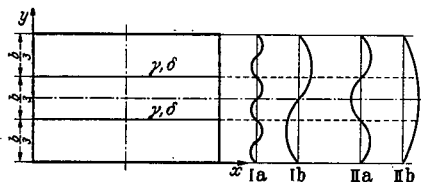
Figure 9.- Buckling forms of plate with two longitudinal stiffeners in equal distances. I-Antisymmetrical II-Symmetrical buckling forms.
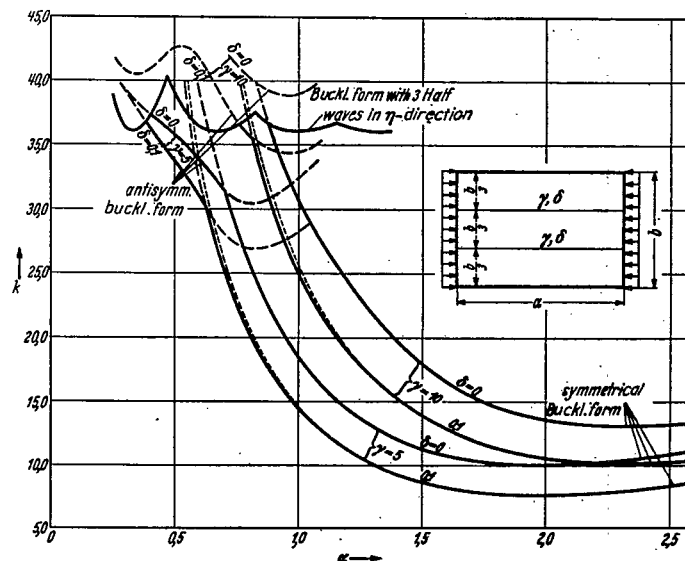
Figure 10.- Buckling stresses of plate with two longitudinal stiffeners at equal distances. The dotted lines for symmetrical buckling form(at δ = 0.1)Correspond to Timoshenko's solutions. (All borders hinged).
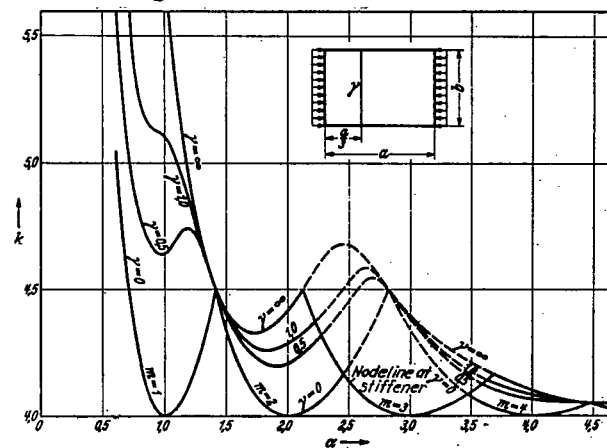

Figure 15.- Buckling stresses of plate with two transversal stiffeners at equal distances.
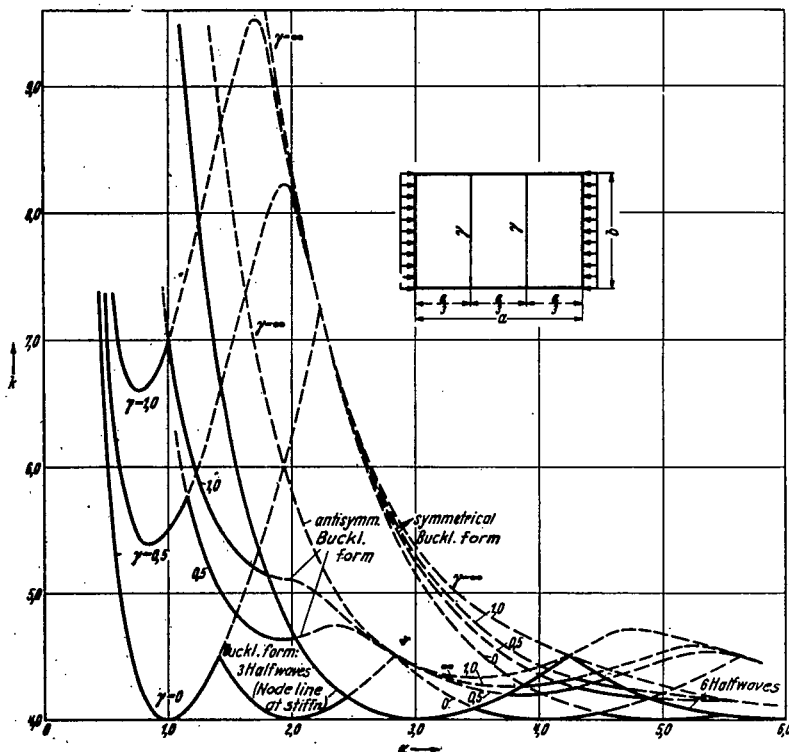

Figure 14.- Buckling stresses of plate with one transversal stiffener at 1/3 of length a.